Exact measures of evaluation in classical natural deduction

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Inria - LS2N

joint w. with Delia Kesner (IRIF)

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Where we lie

Non-Idempotent
Gardner 94 - de Carvalho 07

Intersection
Coppo-Dezani 80

Type Theory

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Type Theory $\rightarrow$ Curry-Howard correspondence

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characterizes:

- normalization
- complexity classes
- MSO-sat.

Type Theory

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Bernadet-Lengrand 11

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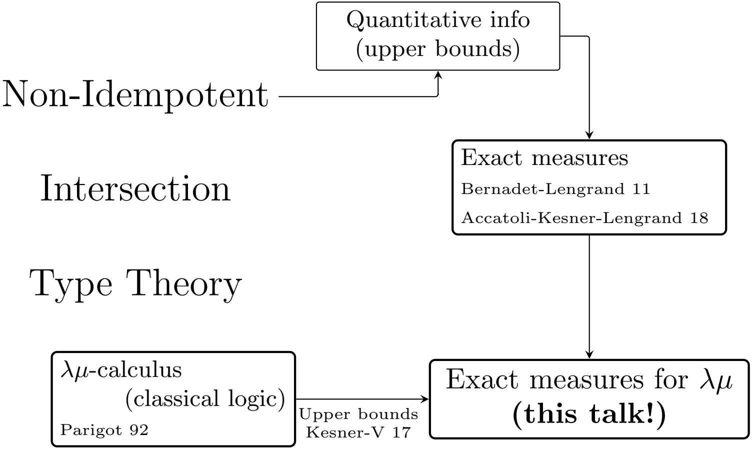
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Exact measures for $\lambda\mu$
(this talk!)

Plan

Intersections types (Coppo, Dezani, 1980)

Goal

Equivalences of the form

"the program t is typable iff it can reach a terminal state"

Idea: **several** certificates to a same subprogram (next slides).

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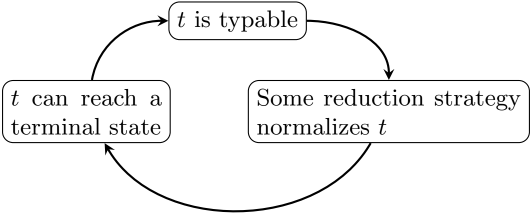

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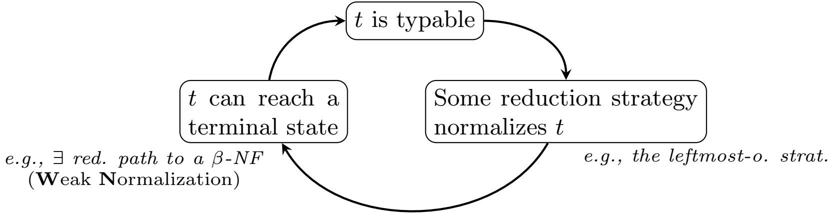

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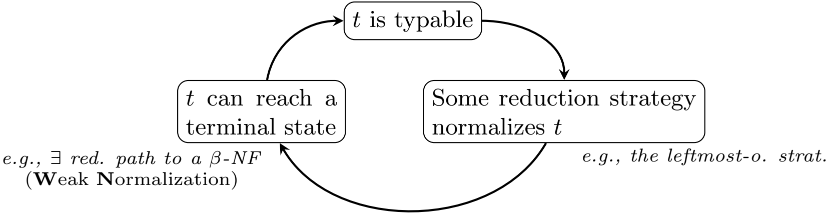

t is WN iff the leftmost-o. stategy terminates on t

nothing to do with types

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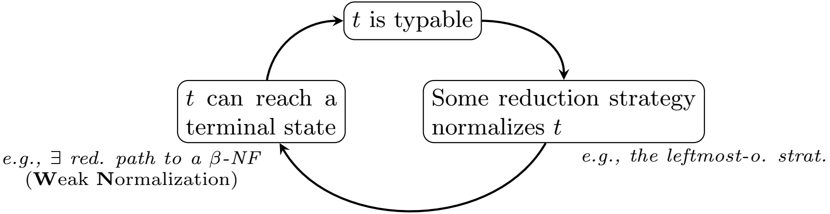

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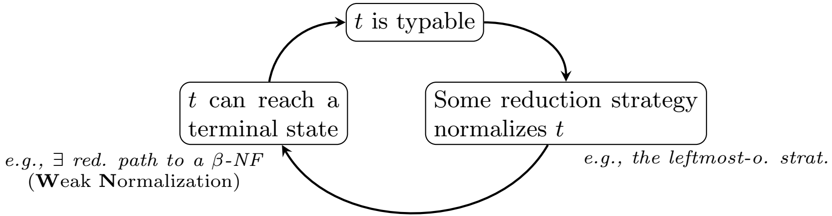

Intersection types

- Perhaps too expressive...
- ...but certify **reduction strategies!**

- Naively, $A \wedge B$ stands for $A \cap B$:

$$\textit{t is of type } A \wedge B \textit{ if } t \textit{ can be typed with } A \textit{ as well as } B.$$

$$\frac{I : A \to A \qquad I : (A \to B) \to (A \to B)}{I : (A \to A) \wedge ((A \to B) \to (A \to B))} \wedge\!-\mathtt{intro} \qquad (\textit{with } I = \lambda x.x)$$

Intuitions (syntax)

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- Intersection = kind of *finite polymorphism*.

$$(A \to A) \wedge ((A \to B) \to (A \to B)) = \textbf{double} \text{ instance of } \forall X.X \to X$$

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- But *less constrained*:

$$\text{assigning } x : o \wedge (o \to o') \wedge (o \to o) \to o \text{ is legal.}$$

$$\textit{(}\boldsymbol{not}\textit{ an instance of a polymorphic type except } \forall X.X := \texttt{False}\textit{!)}$$

Subject Reduction and Subject Expansion

A good intersection type system should enjoy:

Subject Reduction (SR):
Typing is stable under reduction.

Subject Expansion (SE):
Typing is stable under anti-reduction.

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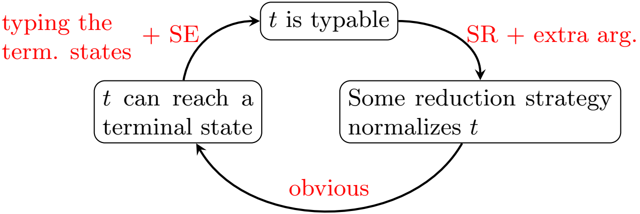

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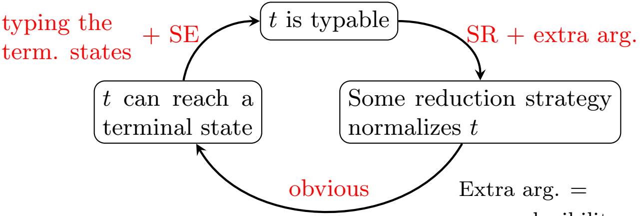

Extra arg. =

- reducibility cand.
- non-trivial well-founded order.
- can it be simpler?

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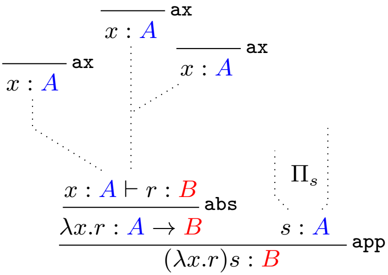

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$$\dfrac{\dfrac{x : A \vdash r : B}{\lambda x.r : A \to B}\,\texttt{abs} \qquad \overset{\Pi_s}{s : A}}{(\lambda x.r)s : B}\,\texttt{app} \qquad \longrightarrow$$

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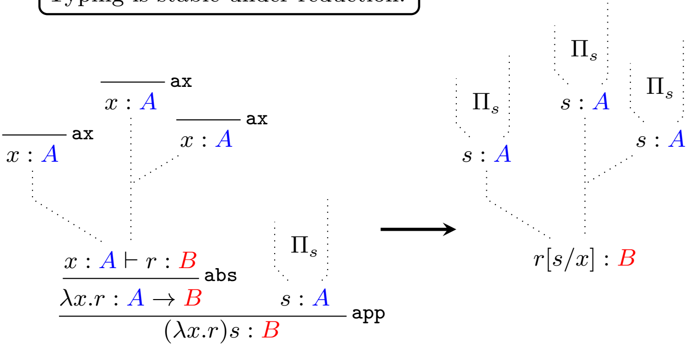

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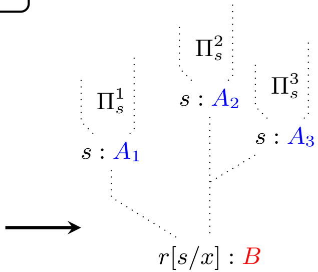

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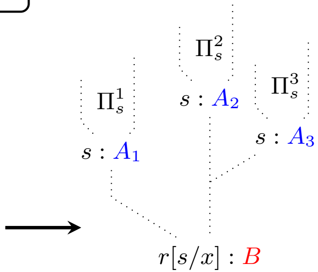

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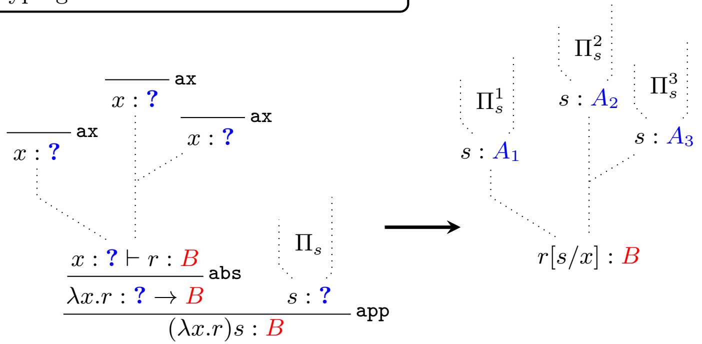

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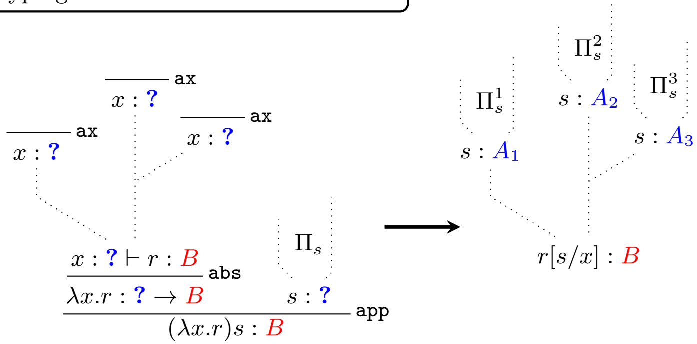

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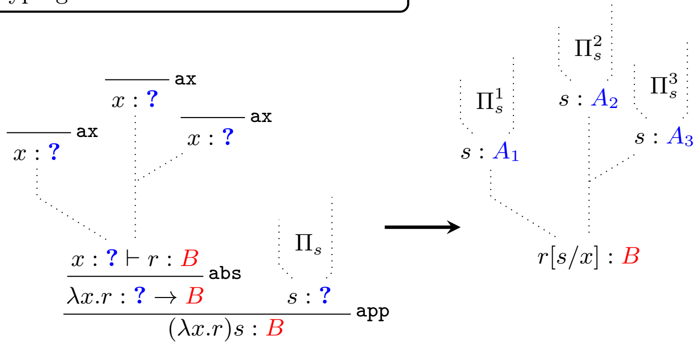

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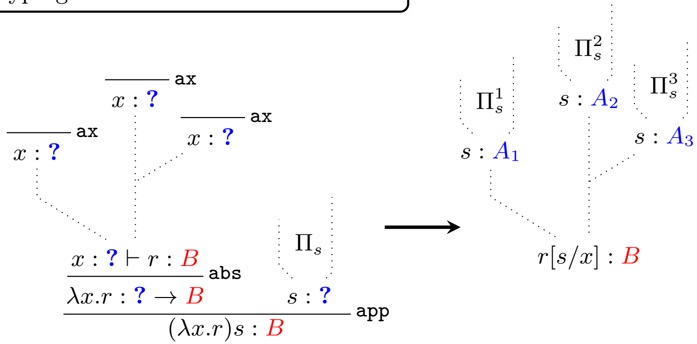

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$$x : A_1 \wedge A_2 \wedge A_3 \vdash x : A_i \ (i = 1, 2, 3)$$

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- Typing normal form: just structural induction (no clash).

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Computation causes **duplication**.

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Non-idempotent intersection types

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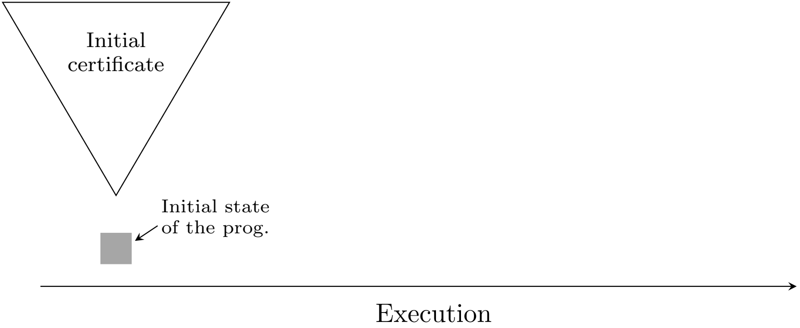

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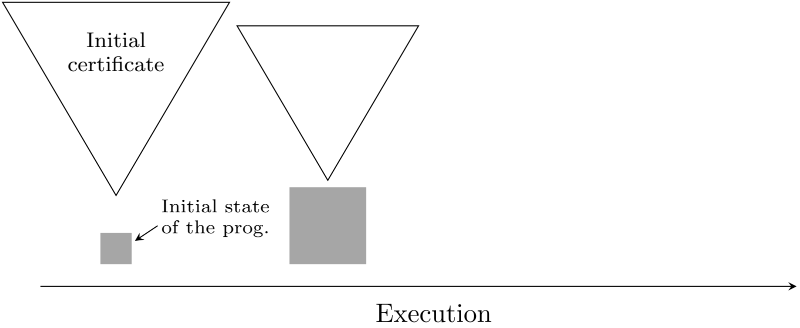

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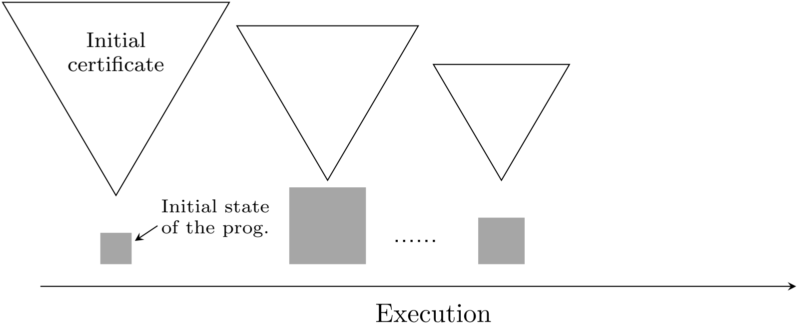

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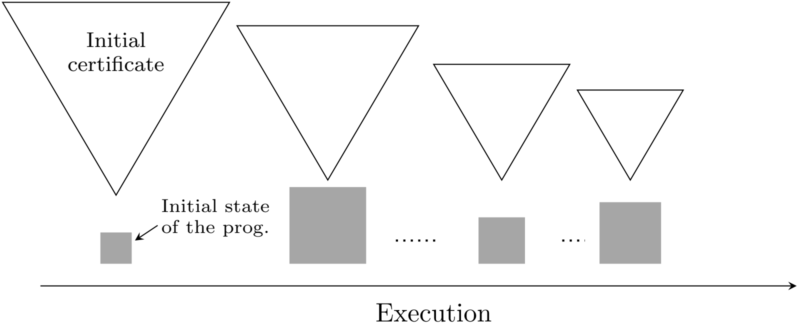

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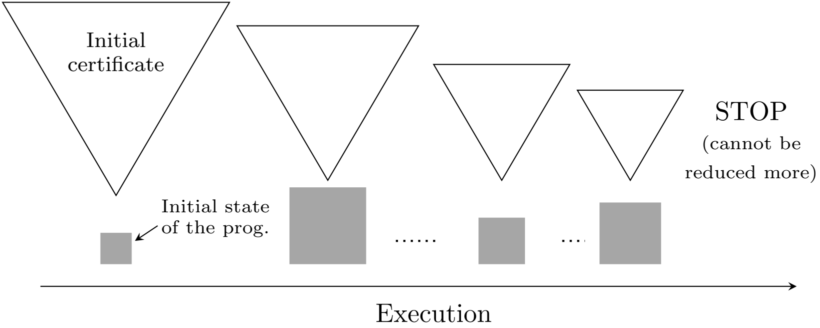

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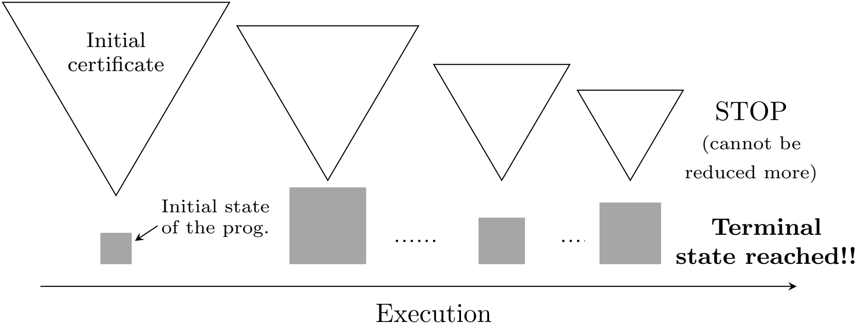

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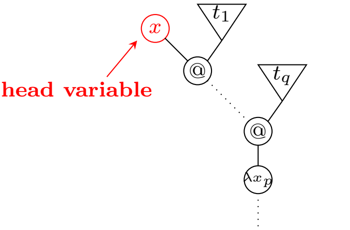
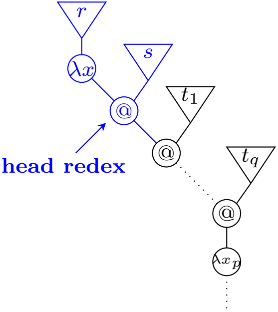
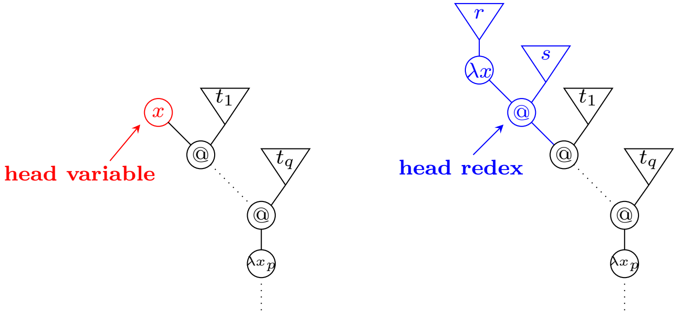
Head Normalization (λ)



Head Normal Form



Head Reducible Term



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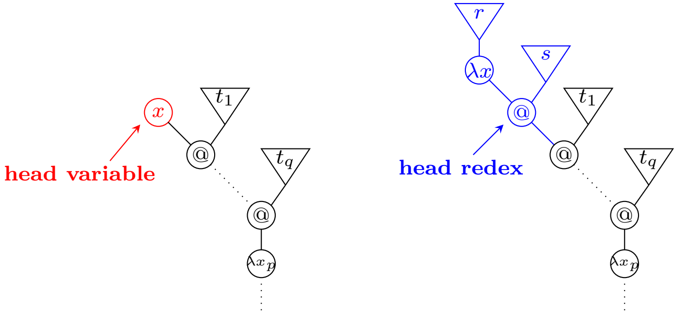

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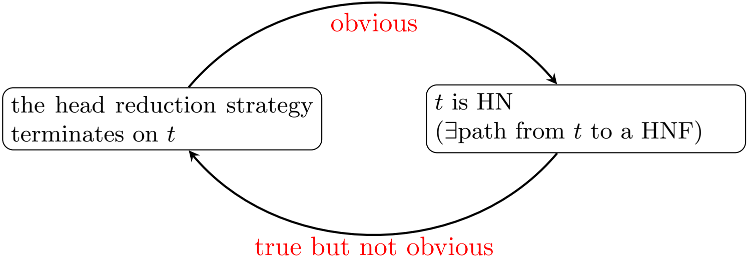

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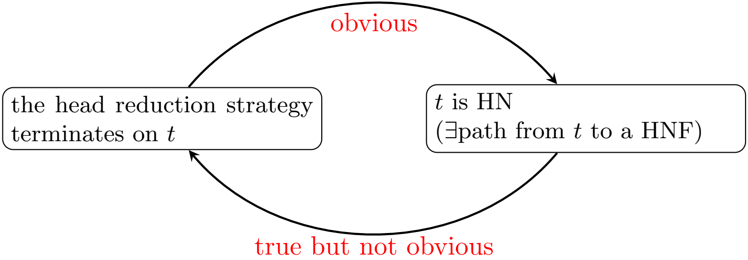

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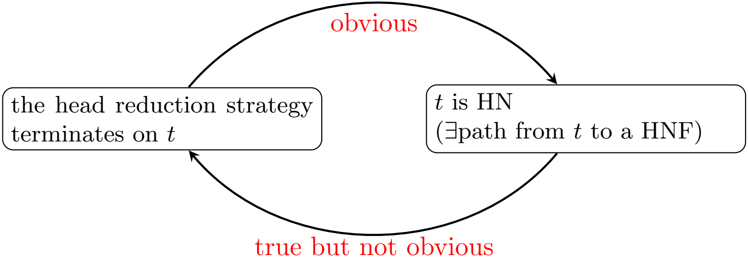

Intersection types come to help!

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- Collapsing $A \wedge B \wedge C$ into $[A, B, C]$ (**multiset**) $\rightsquigarrow$ no need for perm rules etc.

$$A \wedge B \wedge A := [A, B, A] = [A, A, B] \neq [A, B] \qquad [A, B, A] = [A, B] + [A]$$

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- only on the left-h.s of $\to$ (*strictness*)

System $\mathscr{R}_0$ (Gardner 94-de Carvalho 07)

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Remark

- **Relevant** system (no weakening, *cf.* `ax`-rule)
- **Non-idempotency ($\sigma \wedge \sigma \neq \sigma$):**
  in `app`-rule, pointwise multiset sum *e.g.*,

$$(x : [\sigma]; y : [\tau]) + (x : [\sigma, \tau]) = x : [\sigma, \sigma, \tau]; y : [\tau]$$

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Head redexes always typed!

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Head redexes always typed!

but an arg. may be typed 0 time

Properties ($\mathscr{R}_0$)

- **Weighted Subject Reduction**
  - Reduction preserves types and environments, and...
  - ...*head* reduction strictly decreases the number of nodes of the deriv. tree (`size`).

    (actually, holds for any typed redex)
- **Subject Expansion**
  - Anti-reduction preserves types and environments.

Theorem (de Carvalho)

Let t be a λ-term. Then equivalence between:

1. *t is typable (in $\mathscr{R}_0$)*
2. *t is HN*
3. *the head reduction strategy terminates on t ($\rightsquigarrow$ **certification!**)*

Bonus (quantitative information)

If Π types t, then $\texttt{size}(\Pi)$ bounds the number of **steps** of the head red. strategy on t

Let t be a λ-term.

- **Head normalization (HN):**
  there is a path from t to a head normal form.

- **Weak normalization (WN):**
  there is *at least one path* from t to a β**-Normal Form** (NF)

- **Strong normalization (SN):**
  there is *no infinite path* starting at t.

$$\boxed{\mathrm{SN} \Rightarrow \mathrm{WN} \Rightarrow \mathrm{HN}}$$

Nota Bene: $y\,\Omega$ HNF but not WN $\qquad$ $(\lambda x.y)\Omega$ WN but not SN

Characterizing Weak and Strong Normalization

HN	System $\mathscr{R}_0$ $\boxed{\textit{any}\text{ arg. can be left }\textit{untyped}}$	$\mathtt{sz}(\Pi)$ bounds the number of *head* reduction steps
WN	System $\mathscr{R}_0$ + **unforgetfulness criterion** $\boxed{\textit{non-erasable}\text{ args must be typed}}$	$\mathtt{sz}(\Pi)$ bounds the number of leftmost-outermost red. steps (and more)
SN	Modify system $\mathscr{R}_0$ with **choice operator** $\boxed{\textit{all}\text{ args must be typed}}$	$\mathtt{sz}(\Pi)$ bounds the length of *any* reduction path

Subject reduction and expansion in $\mathscr{R}_0$

From a typing of $(\lambda x.r)s$ … to a typing of $r[s/x]$

$$\dfrac{}{x:[\sigma_1] \vdash x:\sigma_1}\,\texttt{ax} \qquad \dfrac{}{x:[\sigma_2] \vdash x:\sigma_2}\,\texttt{ax} \qquad \dfrac{}{x:[\sigma_1] \vdash x:\sigma_1}\,\texttt{ax}$$

$$\dfrac{\dfrac{\Gamma;\ x:[\sigma_1,\sigma_2,\sigma_1] \vdash r:\tau}{\Gamma \vdash \lambda x.r : [\sigma_1,\sigma_2,\sigma_1] \to \tau}\,\texttt{abs} \quad \begin{array}{c}\Pi_1^a \\ \Delta_1^a \vdash s:\sigma_1\end{array} \quad \begin{array}{c}\Pi_2 \\ \Delta_2 \vdash s:\sigma_2\end{array} \quad \begin{array}{c}\Pi_1^b \\ \Delta_1^b \vdash s:\sigma_1\end{array}}{\Gamma + \Delta_1^a + \Delta_1^b + \Delta_2 \vdash (\lambda x.r)s : \tau}\,\texttt{app}$$

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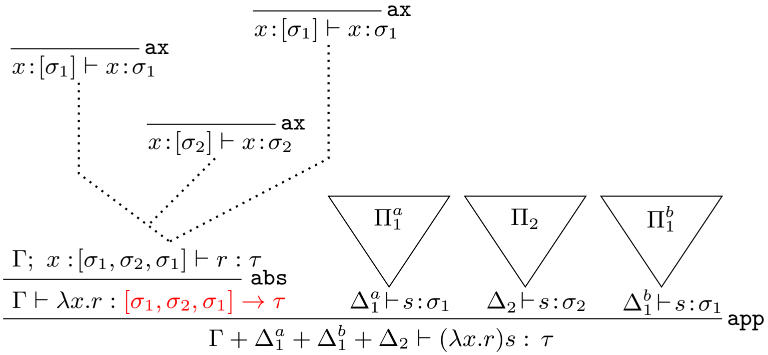

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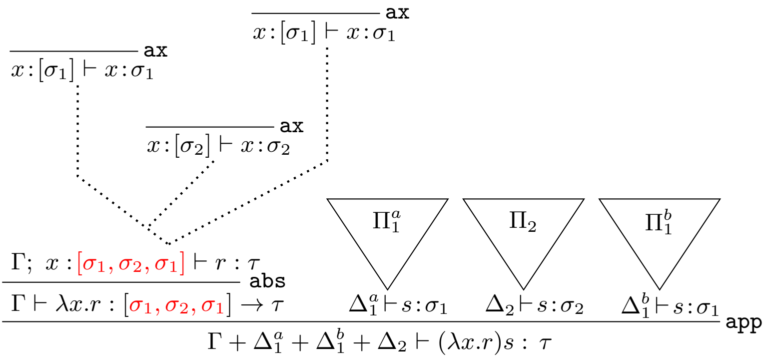

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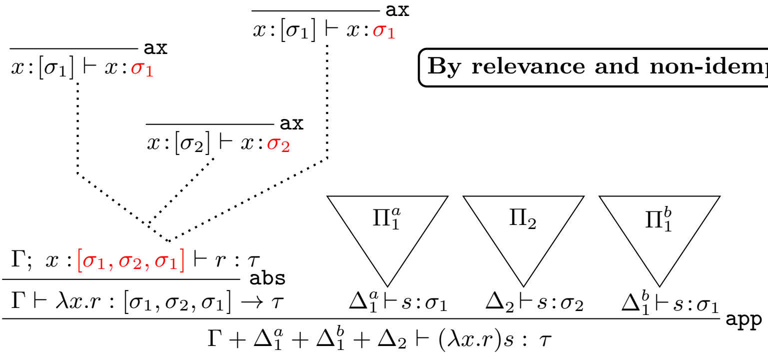
By relevance and non-idempotency!



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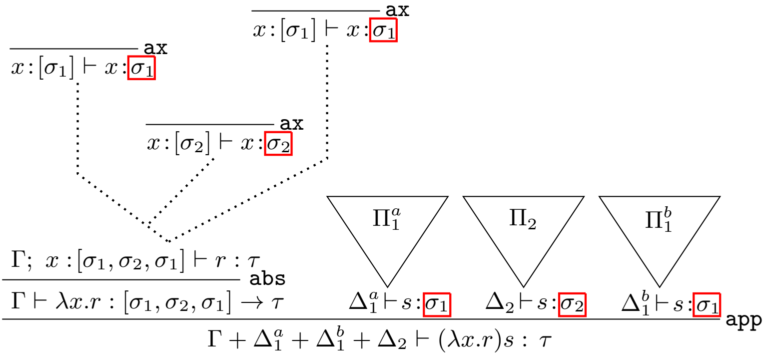

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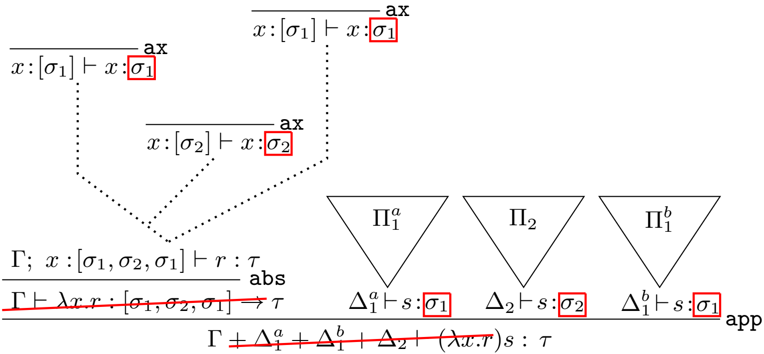

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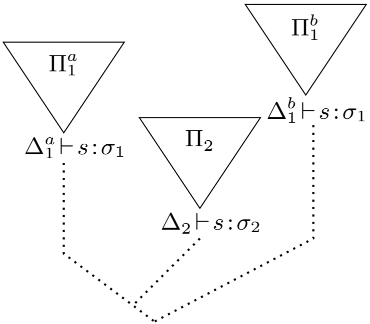

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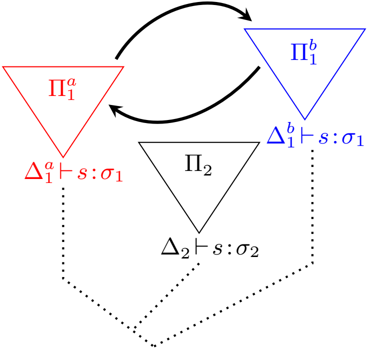

$\boxed{\textbf{Non-determinism of SR}}$

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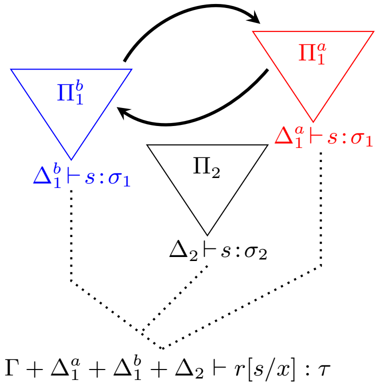

Non-determinism of SR

Plan

Exact measures for the λ-calculus (principles)

- Building on [Accatoli-Kesner-Lengrand,ICFP'18] and [Bernadet-Lengrand,LCMS'13]

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head $\rightarrow_{\mathtt{hd}}$	leftmost-o. $\rightarrow_{\mathtt{lo}}$	maximal $\rightarrow_{\mathtt{mx}}$ *(computes a longest red. path)*

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Remark:

- `hd`-NF=HNF
  v.s. `lo`/`mx`-NF= full NF (no redex).

- $|\lambda x_1 \ldots x_p . x\, t_1 \ldots t_q|_{\mathtt{hd}} = p + q + 1$

New formalism

$\boxed{\begin{array}{l}\bullet\ \textbf{Persistent}\text{ elements} \\ \text{(remain in the NF)} \\ \bullet\ \textbf{Consuming}\text{ elements} \\ \text{(used during red.)}\end{array}}$

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Rules

$$\Gamma \vdash^{({\color{red}\ell},{\color{blue}f})} t : \tau$$

number of β-steps — size of the norm. form

$$\frac{}{x : [\sigma] \vdash^{(0,1)} x : \sigma}\ (\mathtt{ax})$$

$$\frac{\Gamma \vdash^{(\ell,f)} t : \sigma}{\Gamma \setminus\!\!\setminus x \vdash^{(\ell+1, f - \#\Gamma(x))} \lambda x.t : \Gamma(x) \to \sigma}\ (\to_{\mathtt{i}}) \qquad \frac{\Gamma \vdash^{(\ell,f)} t : \mathtt{tight} \quad \mathtt{tight}(\Gamma(x))}{\Gamma \vdash^{(\ell,f+1)} \lambda x.t : \bullet}\ (\bullet_S)$$

$$\frac{\Gamma \vdash^{(\ell_t,f_t)} t : \mathcal{F} \qquad \Delta \Vdash^{(\ell_u,f_u)} u : \mathtt{dom}(\mathcal{F})}{\Gamma \wedge \Delta \vdash^{(\ell_f+\ell_u, f_t+f_u+\#_{\mathtt{p}}\mathcal{F})} t\,u : \mathtt{codom}(\mathcal{F})}\ (\mathtt{app}_S)$$

with $\#_{\mathtt{p}} ? \to ? = 0$ and $\#_{\mathtt{p}} ? \nrightarrow ? = 1$.

Systems $\mathcal{X}^{\lambda}_{\mathtt{hd/lo}}$

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with $\#_{\mathtt{p}}? \to ? = 0$ and $\#_{\mathtt{p}}? \nrightarrow ? = 1$.

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Systems $\mathcal{X}^{\lambda}_{\mathtt{hd/lo}}$

$$\Gamma \vdash^{({\color{red}\ell},{\color{blue}f})} t : \tau$$

number of β-steps   size of the norm. form

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Systems $\mathcal{X}^{\lambda}_{\mathtt{hd/lo}}$

Rules

$$\Gamma \vdash^{(\ell, f)} t : \tau$$

number of β-steps (ℓ) — size of the norm. form (f)

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Systems $\mathcal{X}^{\lambda}_{\mathtt{hd/lo}}$

Auxiliary: $$\frac{(\Gamma_k \vdash^{(\ell_k, f_k)} t : \tau_k)_{k \in K}}{\wedge_{k \in K} \Gamma_k \Vdash^{(+_{k\in K} \ell_k, +_{k \in K} f_k)} t : [\tau_k]_{k \in K}} \,\wedge$$

A simple example

Let $\mathtt{I} = \lambda x.x$.

$$(\lambda x.x\,x)\mathtt{I} \to_S \mathtt{I}\,\mathtt{I} \to_S \mathtt{I} \qquad (S \in \{\mathtt{hd}, \mathtt{lo}, \mathtt{mx}\})$$

Expected counter (2,2)

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Properties of $\mathcal{X}^{\lambda}_{\texttt{hd/lo}}$

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Let $t \in \Lambda$. Then:

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Unique parametrized system

for the 3 strategies

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Plan

1. Overview (idempotent or not intersection types)
2. Non-idempotent intersection types
3. Extracting exact lengths of reduction
4. Resources for Classical Logic
5. Exact Measures for $\lambda\mu$
6. Perspectives

The Lambda-Mu Calculus

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- **Parigot 92**: $\lambda\mu$-calculus = computational interpretation of classical *natural deduction* (*e.g.*, *vs.* $\bar{\lambda}\mu\tilde{\mu}$).
  judg. of the form $A, A \to B \vdash {\color{red}A} \mid {\color{blue}B}, {\color{blue}C}$

$$\dfrac{\dfrac{\dfrac{\dfrac{}{(A \to B) \to A \vdash (A \to B) \to A} \qquad \dfrac{\dfrac{}{A \vdash A, B}}{\vdash A \to B, A}}{(A \to B) \to A \vdash A, A}}{(A \to B) \to A \vdash A}}{\vdash ((A \to B) \to A) \to A}$$

Standard Style

$$
\dfrac{
  \dfrac{}{(A \to B) \to A \vdash (A \to B) \to A}
  \qquad
  \dfrac{\dfrac{}{A \vdash A, \color{red}{B}}}{\vdash A \to B, A}
}{
  \dfrac{(A \to B) \to A \vdash A, A}{
    \dfrac{(A \to B) \to A \vdash A}{\vdash ((A \to B) \to A) \to A}
  }
}
$$

Standard Style

Peirce's Law in Classical Natural Deduction

$$
\dfrac{\dfrac{\dfrac{}{(A \to B) \to A \vdash (A \to B) \to A \mid} \qquad \dfrac{\dfrac{\dfrac{}{A \vdash A \mid B}}{A \vdash B \mid A}}{\vdash A \to B \mid A}\,\texttt{act}}{\dfrac{(A \to B) \to A \vdash A \mid A}{(A \to B) \to A \vdash A \mid}}}{\vdash ((A \to B) \to A) \to A \mid}
$$

Focussed Style

In the right hand-side of $\Gamma \vdash F \mid \Delta$

- 1 active formula F
- inactive formulas Δ

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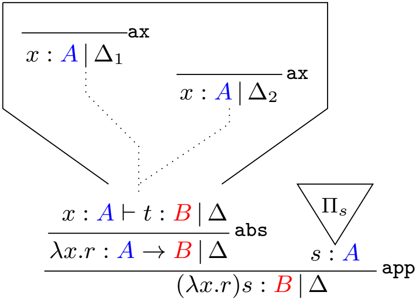
How do we adapt the non-idempotent machinery to $\lambda\mu$?

Cut-elimination steps (classical case)



Cut-elimination steps (classical case)

$$\dfrac{\dfrac{C \mid A \to B}{A \to B \mid C}\,\texttt{act} \qquad \dfrac{\Pi_s}{s : A}}{B \mid C}\,\texttt{app}$$

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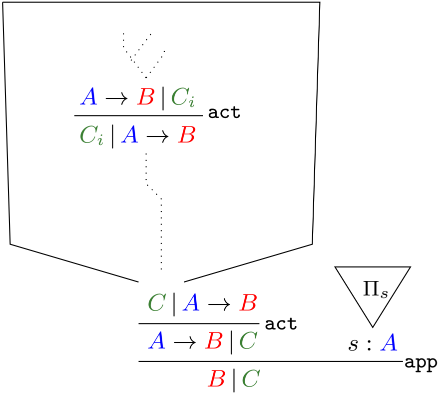

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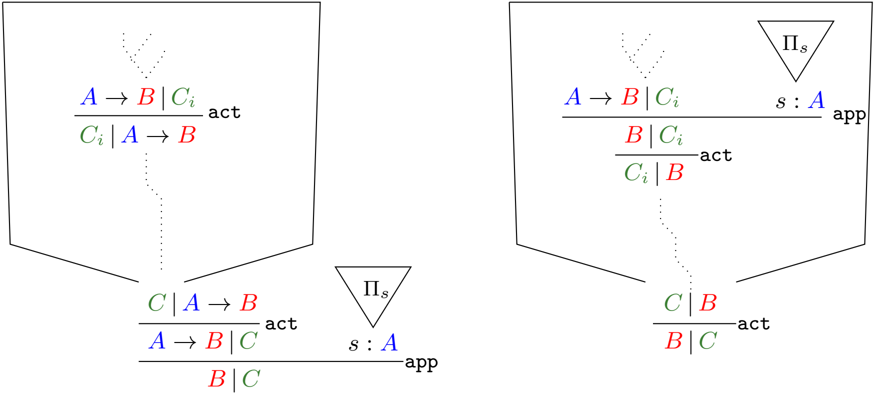

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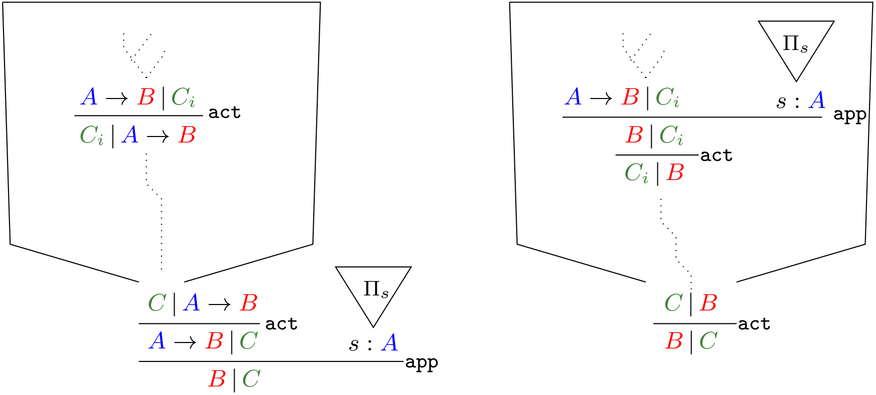

- Duplication of s
- Creation of app-rules
- B saved instead of $A \to B$

The Typing System

Intersection: $\mathcal{I}, \mathcal{J} := [\mathcal{U}_k]_{k \in K}$ $\mathcal{U}, \mathcal{V} =: \langle \sigma_k \rangle_{k \in K}$: **Union**

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$$\boxed{\texttt{call-cc} : [[[A] \to B] \to A] \to \langle A, A \rangle \qquad vs. \qquad ((A \to B) \to A) \to A}$$

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Theorem [Kesner,V.,FSCD17]:
Let t be a $\lambda\mu$-term. Equiv. between:

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- Small-step version.

Plan

$$\frac{\mathcal{U} \neq \langle\,\rangle}{x : [\mathcal{U}] \vdash^{(0,0,1)} x : \mathcal{U} \mid \emptyset}\ (\mathtt{ax}) \qquad \frac{\Gamma \vdash^{(\ell,m,f)} t : \mathcal{U} \mid \Delta}{\Gamma \vdash^{(\ell,m,f)} [\alpha]t \mid \Delta \vee \alpha : \mathcal{U}}\ (\mathtt{c}) \qquad (\wedge)$$

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Capturing exact measures in $\lambda\mu$

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- $\texttt{rob}(\mathcal{U})$ = counts top-level $\not\rightarrow$ *(= number of future @-nodes)*
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Parametrization

- Parametrized system (tightness + domains)

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(spec. normal forms)

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(spec. if erasable args are typed)

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Theorem (Kesner,V)

let $S \in \{\mathtt{hd}, \mathtt{lo}, \mathtt{mx}\}$ and t a $\lambda\mu$-term. Then:

$t \to_S^{(\ell,m)} t'$ a S-NF with $|t'|_S = f$

iff $\Gamma \vdash^{(\ell,m,f)} t : \mathcal{U} \mid \Delta$ tight for some Γ, $\mathcal{U}$, Δ

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Bonus: completely factorized proofs!

Plan

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forbid duplication of typing deriv.

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Future work:
$\bar{\lambda}\mu\tilde{\mu}$ (seq. calc.)? Call-by-push-value?

Thank you for your attention!