Consuming and Persistent Types for Classical Logic

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Summary

Intersection types (Coppo-Dezani 80)

t typable **iff** t terminates

↳ *i.e. typability charac. termination*

non-idempotency (Gardner 94 - Carvalho 07)

$\rightsquigarrow$ · **quantitative** info. (**upper bounds**)

· simple proofs of termin.

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$\rightsquigarrow$ *eval. length + size of the n.f.*

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The $\lambda\mu$-calculus (Parigot 92)

- computational interpretation of **classical** natural deduction.
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- β-red. + **μ-red.**
- Judgments of the form:

$$\Gamma \vdash t : \mathcal{U} \mid \Delta$$

types variables — types co-variables

"names"

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A type system such that, for all $\lambda\mu$-terms t, t evaluates to normal form t' of size f in ℓ β-steps and m μ-steps

iff $\Gamma \vdash^{(\ell,m,f)} t : \mathcal{U} \mid \Delta$

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For **3 eval.** & normalizations
· **head** eval. & head norm.
· **left.-outer.** eval. & weak norm.
· **max.** eval. & strong norm.

Parametrized approach

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ex: $(\lambda x.y @ x @ x) @ z \to_\beta y @ z @ z$

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- One type constant $\bullet$ (meaning "not applied")

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Solution

use *non-idempotent* types

$\rightsquigarrow$ linearize terms

lin. = create copies of args. *before* they are duplicated

ex: $(\lambda x.x\,x)[\mathtt{I},\mathtt{I}] \to_\beta \mathtt{I}\,\mathtt{I} \to_\beta \mathtt{I}$

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Dealing with control operators

New case of *redex creation*

$\rightsquigarrow$ "activate" persistent arrows into consuming ones

app. constructors are created by μ-red.

Plan

1. The Lambda-Mu Calculus
2. Non-idempotent intersection types
3. Capturing exact measures (length + normal form)

- Intuit. logic + **Peirce's Law** $((A \to B) \to A) \to A$
  gives **classical logic**.

- **Griffin 90**: `call–cc` and Felleisen's $\mathcal{C}$-operator typable with Peirce's Law $((A \to B) \to A) \to A$
  $\leadsto$ the **Curry-Howard** iso extends to classical logic

$$\boxed{\textbf{classical logic}} \longleftrightarrow \boxed{\textbf{backtracking}}$$

- **Parigot 92**: $\lambda\mu$-calculus
  = computational interpretation of classical *natural deduction* $(\neq \bar{\lambda}\mu\tilde{\mu})$

  judgement form: $A, A \to B \vdash A \mid B, C$

$$\dfrac{\dfrac{\dfrac{\dfrac{}{(A \to B) \to A \vdash (A \to B) \to A} \qquad \dfrac{\dfrac{}{A \vdash A, B}}{\vdash A \to B, A}}{(A \to B) \to A \vdash A, A}}{(A \to B) \to A \vdash A}}{\vdash ((A \to B) \to A) \to A}$$

Standard Style

$$\dfrac{\dfrac{\dfrac{}{(A \to B) \to A \vdash (A \to B) \to A} \qquad \dfrac{\dfrac{}{A \vdash A, {\color{red}B}}}{\vdash A \to B, A}}{(A \to B) \to A \vdash A, A}}{\dfrac{(A \to B) \to A \vdash A}{\vdash ((A \to B) \to A) \to A}}$$

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Peirce's Law in Classical Natural Deduction

$$\dfrac{\dfrac{\dfrac{\dfrac{}{(A \to B) \to A \vdash (A \to B) \to A \mid} \qquad \dfrac{\dfrac{\dfrac{}{A \vdash A \mid B}}{A \vdash B \mid A}\,\texttt{act}}{\vdash A \to B \mid A}}{(A \to B) \to A \vdash A \mid A}}{(A \to B) \to A \vdash A \mid}}{\vdash ((A \to B) \to A) \to A \mid}$$

Focussed Style

In the right hand-side of $\Gamma \vdash F \mid \Delta$

- 1 active formula F
- inactive formulas Δ

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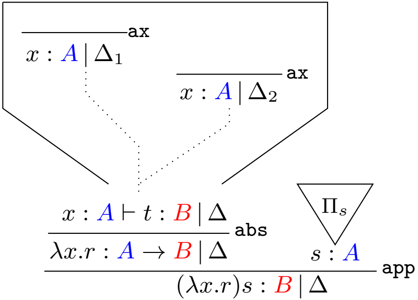
μ**-red**: duplication + creation of app.



As usual...

β-step

rules `abs` + `app` (*i.e.* intro + elim)

Cut-elimination steps (classical case)

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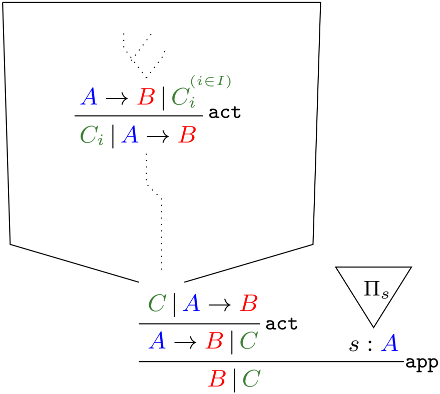
$$\dfrac{\dfrac{C \mid A \to B}{A \to B \mid C}\,\texttt{act} \qquad \dfrac{\Pi_s}{s : A}}{B \mid C}\,\texttt{app}$$

μ-step
activ. of an *arrow* type + app

which has been deact. before



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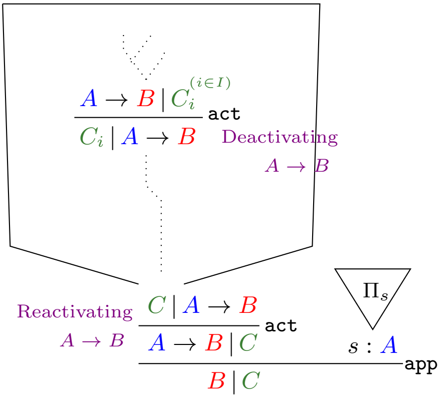

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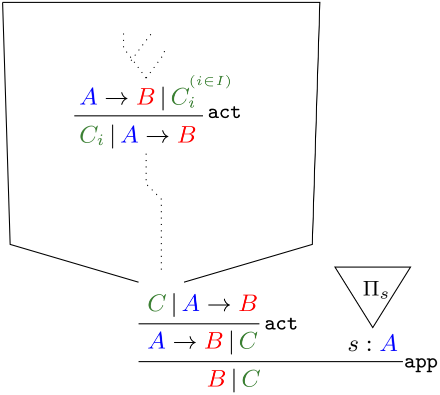

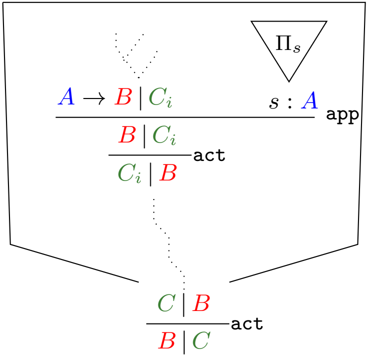

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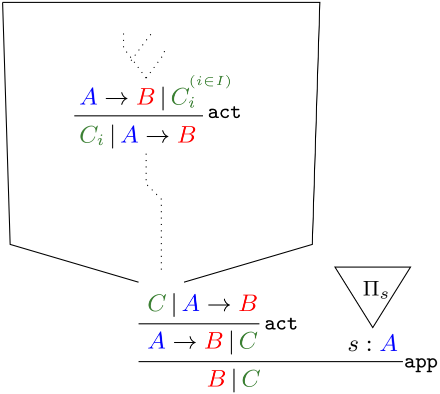

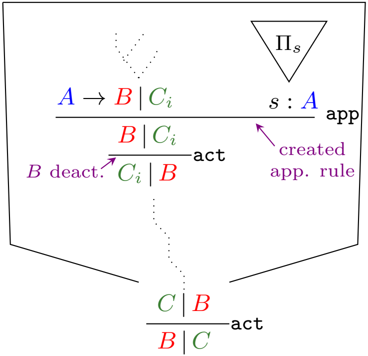

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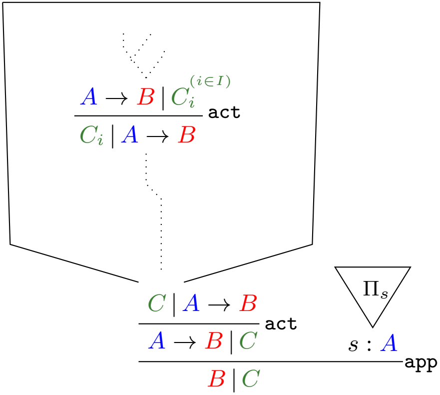

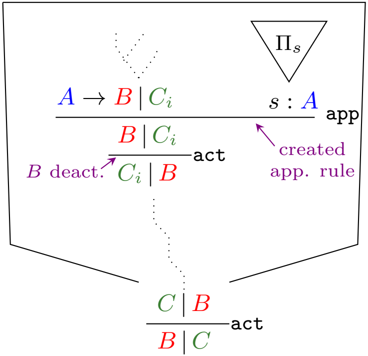

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- Duplication of s
- Creation of app-rules
- B saved instead of $A \to B$

Plan

Non-idempotent intersection (example)

$\mathtt{I} := \lambda y.y$

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- **blue:** persistent
- **red:** consuming

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Principles of intersection types

- **Intersection:** $x : A \cap B$
  $\rightsquigarrow$ x has types A and B simultaneously
- **Non-idem. setting:** $x : A \cap B \cap A$
  $\rightsquigarrow$ x has type A twice and type B *once*
  $\rightsquigarrow$ one write $x : [A, B, A]$ (**multiset**)

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not simply typable   simply typable

$\boxed{F := o \to o}$

$$\dfrac{\mathtt{I} : F \to F \qquad \dfrac{\mathtt{I} : F \to F \qquad \mathtt{I} : F}{\mathtt{I}\,\mathtt{I} : F}}{\mathtt{I}(\mathtt{I}\,\mathtt{I}) : F}$$

Typing $\mathtt{I}(\mathtt{I}\,\mathtt{I})$

x may be assigned *several* types
$\rightsquigarrow$ x **placeholder** for $\mathtt{I}$

$$\dfrac{\dfrac{}{x : F \to F} \qquad \dfrac{\dfrac{}{x : F \to F} \qquad \dfrac{}{x : F}}{x\,x : F}}{x(x\,x) : F}$$

$\mathtt{I} := \lambda y.y$

Let us type

$\boxed{(\lambda x.x(x\,x))\mathtt{I}} \to_{\mathtt{h}} \mathtt{I}(\mathtt{I}\,\mathtt{I}) \to_{\mathtt{h}} \mathtt{I}\,\mathtt{I} \to_{\mathtt{h}} \mathtt{I}$

not simply typable — simply typable

$$\boxed{F := o \to o}$$

$$\dfrac{\mathtt{I} : F\to F \qquad \dfrac{\mathtt{I} : F\to F \qquad \mathtt{I} : F}{\mathtt{I}\,\mathtt{I} : F}}{\mathtt{I}(\mathtt{I}\,\mathtt{I}) : F}$$

Typing $\mathtt{I}(\mathtt{I}\,\mathtt{I})$

$$\dfrac{\dfrac{}{x : F \to F} \qquad \dfrac{\dfrac{}{x : F \to F} \qquad \dfrac{}{x : F}}{x\,x : F}}{x(x\,x) : F}$$

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simply typable

$$\boxed{F := o \to o}$$

$$\dfrac{\mathtt{I} : F \to F \qquad \dfrac{\mathtt{I} : F \to F \qquad \mathtt{I} : F}{\mathtt{I}\,\mathtt{I} : F}}{\mathtt{I}(\mathtt{I}\,\mathtt{I}) : F}$$

Typing $\mathtt{I}(\mathtt{I}\,\mathtt{I})$

Environment:

$$\dfrac{\dfrac{}{x : F \to F} \qquad \dfrac{\dfrac{}{x : F \to F} \qquad \dfrac{}{x : F}}{x\,x : F}}{x(x\,x) : F}$$

Non-idempotent intersection (example)

$\mathtt{I} := \lambda y.y$

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not simply typable

simply typable

$\boxed{F := o \to o}$

$$\dfrac{\mathtt{I} : F \to F \qquad \dfrac{\mathtt{I} : F \to F \qquad \mathtt{I} : F}{\mathtt{I}\,\mathtt{I} : F}}{\mathtt{I}(\mathtt{I}\,\mathtt{I}) : F}$$

Typing $\mathtt{I}(\mathtt{I}\,\mathtt{I})$

Environment:
$x : [\, F \to F, F] \vdash \ldots$

$$\dfrac{\dfrac{}{x : F \to F} \qquad \dfrac{\dfrac{}{x : \boxed{F \to F}} \qquad \dfrac{}{x : \boxed{F}}}{x\,x : F}}{x(x\,x) : F}$$

$\mathtt{I} := \lambda y.y$

Let us type

$\boxed{(\lambda x.x(x\,x))\mathtt{I}} \rightarrow_{\mathtt{h}} \mathtt{I}(\mathtt{I}\,\mathtt{I}) \rightarrow_{\mathtt{h}} \mathtt{I}\,\mathtt{I} \rightarrow_{\mathtt{h}} \mathtt{I}$

not simply typable

simply typable

$$\boxed{F := o \rightarrow o}$$

$[F \rightarrow F, F] = (F \rightarrow F) \cap F$

multiset $\leftrightarrow$ inter.

Environment:
$x : [F \rightarrow F, F] \vdash \ldots$

$$\dfrac{\dfrac{}{x : F \rightarrow F} \qquad \dfrac{\dfrac{}{x : F \rightarrow F} \quad \dfrac{}{x : F}}{x\,x : F}}{x(x\,x) : F}$$

$$\dfrac{\mathtt{I} : F \rightarrow F \qquad \dfrac{\mathtt{I} : F \rightarrow F \quad \mathtt{I} : F}{\mathtt{I}\,\mathtt{I} : F}}{\mathtt{I}(\mathtt{I}\,\mathtt{I}) : F}$$

Typing $\mathtt{I}(\mathtt{I}\,\mathtt{I})$

Non-idempotent intersection (example)

$\mathtt{I} := \lambda y.y$

Let us type

$\boxed{(\lambda x.x(x\,x))\mathtt{I}} \to_{\mathtt{h}} \mathtt{I}(\mathtt{I}\,\mathtt{I}) \to_{\mathtt{h}} \mathtt{I}\,\mathtt{I} \to_{\mathtt{h}} \mathtt{I}$

not simply typable

simply typable

$$\boxed{F := o \to o}$$

$$\dfrac{\mathtt{I} : F \to F \qquad \dfrac{\mathtt{I} : F \to F \qquad \mathtt{I} : F}{\mathtt{I}\,\mathtt{I} : F}}{\mathtt{I}(\mathtt{I}\,\mathtt{I}) : F}$$

Typing $\mathtt{I}(\mathtt{I}\,\mathtt{I})$

Environment:

$$\dfrac{\dfrac{}{x : F \to F} \qquad \dfrac{\dfrac{}{x : F \to F} \qquad \dfrac{}{x : F}}{x\,x : F}}{x(x\,x) : F}$$

$\mathtt{I} := \lambda y.y$

Let us type

$\boxed{(\lambda x.x(x\,x))\mathtt{I}} \to_{\mathtt{h}} \mathtt{I}(\mathtt{I}\,\mathtt{I}) \to_{\mathtt{h}} \mathtt{I}\,\mathtt{I} \to_{\mathtt{h}} \mathtt{I}$

not simply typable

simply typable

$$\boxed{F := o \to o}$$

$$\dfrac{\mathtt{I} : F \to F \qquad \dfrac{\mathtt{I} : F \to F \qquad \mathtt{I} : F}{\mathtt{I}\,\mathtt{I} : F}}{\mathtt{I}(\mathtt{I}\,\mathtt{I}) : F}$$

Typing $\mathtt{I}(\mathtt{I}\,\mathtt{I})$

Environment:
$x : [\,F \to F,\; F \to F, F] \vdash \ldots$

$$\dfrac{\dfrac{}{x : \boxed{F \to F}} \qquad \dfrac{\dfrac{}{x : \boxed{F \to F}} \qquad \dfrac{}{x : \boxed{F}}}{x\,x : F}}{x(x\,x) : F}$$

$\mathtt{I} := \lambda y.y$

Let us type

$\boxed{(\lambda x.x(x\,x))\mathtt{I}} \rightarrow_{\mathtt{h}} \mathtt{I}(\mathtt{I}\,\mathtt{I}) \rightarrow_{\mathtt{h}} \mathtt{I}\,\mathtt{I} \rightarrow_{\mathtt{h}} \mathtt{I}$

not simply typable

simply typable

$$\boxed{F := o \rightarrow o}$$

$$\dfrac{\mathtt{I} : F \rightarrow F \qquad \dfrac{\mathtt{I} : F \rightarrow F \qquad \mathtt{I} : F}{\mathtt{I}\,\mathtt{I} : F}}{\mathtt{I}(\mathtt{I}\,\mathtt{I}) : F}$$

Typing $\mathtt{I}(\mathtt{I}\,\mathtt{I})$

Environment:
$x : [\, F \rightarrow F,\ F \rightarrow F, F] \vdash \ldots$

$$\dfrac{\dfrac{\dfrac{}{x : F \rightarrow F} \qquad \dfrac{\dfrac{}{x : F \rightarrow F} \qquad \dfrac{}{x : F}}{x\,x : F}}{x(x\,x) : F}}{\lambda x.x(x\,x) : [\, F \rightarrow F,\ F \rightarrow F, F] \rightarrow F}$$

Non-idempotent intersection (example)

$\mathtt{I} := \lambda y.y$

Let us type

$$\underbrace{\boxed{(\lambda x.x(x\,x))\mathtt{I}}}_{\text{not simply typable}} \to_{\mathtt{h}} \underbrace{\mathtt{I}(\mathtt{I}\,\mathtt{I}) \to_{\mathtt{h}} \mathtt{I}\,\mathtt{I} \to_{\mathtt{h}} \mathtt{I}}_{\text{simply typable}}$$

$$\boxed{F := o \to o}$$

$$\dfrac{\mathtt{I} : F\to F \qquad \dfrac{\mathtt{I} : F\to F \qquad \mathtt{I} : F}{\mathtt{I}\,\mathtt{I} : F}}{\mathtt{I}(\mathtt{I}\,\mathtt{I}) : F}$$

Typing $\mathtt{I}(\mathtt{I}\,\mathtt{I})$

Environment:
$x : [\,F \to F,\; F \to F, F] \vdash \ldots$

$$\dfrac{\dfrac{\dfrac{}{x : F \to F} \qquad \dfrac{\dfrac{}{x : F \to F} \quad \dfrac{}{x : F}}{x\,x : F}}{x(x\,x) : F}}{\lambda x.x(x\,x) : [\,F \to F,\; F \to F, F] \to F}$$

Quantitative information!
x typed *twice* with $F \to F$
once with F

$\mathtt{I} := \lambda y.y$

Let us type

$$\boxed{(\lambda x.x(x\,x))\mathtt{I}} \to_{\mathtt{h}} \mathtt{I}(\mathtt{I}\,\mathtt{I}) \to_{\mathtt{h}} \mathtt{I}\,\mathtt{I} \to_{\mathtt{h}} \mathtt{I}$$

not simply typable   simply typable

$$\boxed{F := o \to o}$$

$$\dfrac{\mathtt{I} : F \to F \qquad \dfrac{\mathtt{I} : F \to F \qquad \mathtt{I} : F}{\mathtt{I}\,\mathtt{I} : F}}{\mathtt{I}(\mathtt{I}\,\mathtt{I}) : F}$$

Typing $\mathtt{I}(\mathtt{I}\,\mathtt{I})$

$$\dfrac{\dfrac{\dfrac{}{x : F \to F} \qquad \dfrac{\dfrac{}{x : F \to F} \qquad \dfrac{}{x : F}}{x\,x : F}}{x(x\,x) : F}}{\lambda x.x(x\,x) : [\,F \to F,\ F \to F, F] \to F}$$

Non-idempotent intersection (example)

$\mathtt{I} := \lambda y.y$

Let us type

$\boxed{(\lambda x.x(x\,x))\mathtt{I}} \to_{\mathtt{h}} \mathtt{I}(\mathtt{I}\,\mathtt{I}) \to_{\mathtt{h}} \mathtt{I}\,\mathtt{I} \to_{\mathtt{h}} \mathtt{I}$

not simply typable ⤴   simply typable

$$\boxed{F := o \to o}$$

$$\dfrac{\mathtt{I} : F \to F \qquad \dfrac{\mathtt{I} : F \to F \qquad \mathtt{I} : F}{\mathtt{I}\,\mathtt{I} : F}}{\mathtt{I}(\mathtt{I}\,\mathtt{I}) : F}$$

Typing $\mathtt{I}(\mathtt{I}\,\mathtt{I})$

$$\dfrac{\dfrac{}{x : \boxed{F \to F}} \qquad \dfrac{\dfrac{}{x : \boxed{F \to F}} \qquad \dfrac{}{x : \boxed{F}}}{x\,x : F}}{\dfrac{x(x\,x) : F}{\lambda x.x(x\,x) : [\underbrace{F \to F,\ F \to F, F}]\to F}}$$

Quantitative typing
3 types in the domain
$\rightsquigarrow$ I should be typed 3 times

Non-idempotent intersection (example)

$\mathtt{I} := \lambda y.y$

Let us type

$\boxed{(\lambda x.x(x\,x))\mathtt{I}} \rightarrow_{\mathtt{h}} \mathtt{I}(\mathtt{I}\,\mathtt{I}) \rightarrow_{\mathtt{h}} \mathtt{I}\,\mathtt{I} \rightarrow_{\mathtt{h}} \mathtt{I}$

not simply typable — simply typable

$$\boxed{F := o \rightarrow o}$$

$$\dfrac{\mathtt{I} : F \rightarrow F \qquad \dfrac{\mathtt{I} : F \rightarrow F \qquad \mathtt{I} : F}{\mathtt{I}\,\mathtt{I} : F}}{\mathtt{I}(\mathtt{I}\,\mathtt{I}) : F}$$

Typing $\mathtt{I}(\mathtt{I}\,\mathtt{I})$

$$\dfrac{\dfrac{\dfrac{}{x : F \rightarrow F} \qquad \dfrac{\dfrac{}{x : F \rightarrow F} \quad \dfrac{}{x : F}}{x\,x : F}}{x(x\,x) : F}}{\lambda x.x(x\,x) : [\underbrace{F \rightarrow F,\ F \rightarrow F, F}]\rightarrow F}$$

$$\dfrac{\dfrac{}{y : F}}{\mathtt{I} : F \rightarrow F} \qquad \dfrac{\dfrac{}{y : F}}{\mathtt{I} : F \rightarrow F} \qquad \dfrac{\dfrac{}{y : o}}{\mathtt{I} : F \ (= o \rightarrow o)}$$

$\lambda y.y$

Quantitative typing

3 types in the domain

$\rightsquigarrow$ $\mathtt{I}$ should be typed 3 times

Non-idempotent intersection (example)

$\mathtt{I} := \lambda y.y$

Let us type

$\boxed{(\lambda x.x(x\,x))\mathtt{I}} \rightarrow_{\mathtt{h}} \mathtt{I}(\mathtt{I}\,\mathtt{I}) \rightarrow_{\mathtt{h}} \mathtt{I}\,\mathtt{I} \rightarrow_{\mathtt{h}} \mathtt{I}$

not simply typable

simply typable

$$\boxed{F := o \rightarrow o}$$

$$\dfrac{\mathtt{I} : F \rightarrow F \qquad \dfrac{\mathtt{I} : F \rightarrow F \qquad \mathtt{I} : F}{\mathtt{I}\,\mathtt{I} : F}}{\mathtt{I}(\mathtt{I}\,\mathtt{I}) : F}$$

Typing $\mathtt{I}(\mathtt{I}\,\mathtt{I})$

$$\dfrac{\dfrac{\dfrac{}{x : F \rightarrow F} \qquad \dfrac{\dfrac{}{x : F \rightarrow F} \qquad \dfrac{}{x : F}}{x\,x : F}}{x(x\,x) : F}}{\lambda x.x(x\,x) : [F \rightarrow F,\ F \rightarrow F, F] \rightarrow F}$$

$$\dfrac{\dfrac{}{y : F}}{\mathtt{I} : F \rightarrow F} \qquad \dfrac{\dfrac{}{y : F}}{\mathtt{I} : F \rightarrow F} \qquad \dfrac{\dfrac{}{y : o}}{\mathtt{I} : F} \ (= o \rightarrow o)$$

$\mathtt{I} := \lambda y.y$

Let us type

$\boxed{(\lambda x.x(x\,x))\mathtt{I}} \to_{\mathtt{h}} \mathtt{I}(\mathtt{I}\,\mathtt{I}) \to_{\mathtt{h}} \mathtt{I}\,\mathtt{I} \to_{\mathtt{h}} \mathtt{I}$

not simply typable

simply typable

$$\boxed{F := o \to o}$$

$$\dfrac{\mathtt{I} : F \to F \qquad \dfrac{\mathtt{I} : F \to F \qquad \mathtt{I} : F}{\mathtt{I}\,\mathtt{I} : F}}{\mathtt{I}(\mathtt{I}\,\mathtt{I}) : F}$$

Typing $\mathtt{I}(\mathtt{I}\,\mathtt{I})$

$$\dfrac{\dfrac{\dfrac{\dfrac{}{x : F \to F} \qquad \dfrac{\dfrac{}{x : F \to F} \quad \dfrac{}{x : F}}{x\,x : F}}{x(x\,x) : F}}{\lambda x.x(x\,x) : [\,F \to F,\ F \to F, F] \to F} \qquad \dfrac{\dfrac{}{y : F}}{\mathtt{I} : F \to F} \qquad \dfrac{\dfrac{}{y : F}}{\mathtt{I} : F \to F} \qquad \dfrac{\dfrac{}{y : o}}{\mathtt{I} : F \ (= o \to o)}}{(\lambda x.x(x\,x))\mathtt{I} : F}$$

$\mathtt{I} := \lambda y.y$

Let us type

$\boxed{(\lambda x.x(x\,x))\mathtt{I}} \to_{\mathtt{h}} \mathtt{I}(\mathtt{I}\,\mathtt{I}) \to_{\mathtt{h}} \mathtt{I}\,\mathtt{I} \to_{\mathtt{h}} \mathtt{I}$

not simply typable — simply typable

$$\boxed{F := \; o \to o}$$

$$\dfrac{\mathtt{I} : F \to F \qquad \dfrac{\mathtt{I} : F \to F \qquad \mathtt{I} : F}{\mathtt{I}\,\mathtt{I} : F}}{\mathtt{I}(\mathtt{I}\,\mathtt{I}) : F}$$

Typing $\mathtt{I}(\mathtt{I}\,\mathtt{I})$

$$\dfrac{\dfrac{\dfrac{\dfrac{}{x : F \to F} \qquad \dfrac{\dfrac{}{x : F \to F} \quad \dfrac{}{x : F}}{x\,x : F}}{x(x\,x) : F}}{\lambda x.x(x\,x) : [\, F \to F,\; F \to F, F] \to F} \qquad \dfrac{\dfrac{}{y : F}}{\mathtt{I} : F \to F} \qquad \dfrac{\dfrac{}{y : F}}{\mathtt{I} : F \to F} \qquad \dfrac{\dfrac{}{y : o}}{\mathtt{I} : F \; (= o \to o)}}{(\lambda x.x(x\,x))\mathtt{I} : F}$$

Subject expansion works
because x has been assigned *several* types

Subj. exp.: typing stable under anti-reduction

$$\mathtt{I} := \lambda y.y$$

Let us type

$$\boxed{(\lambda x.x(x\,x))\mathtt{I}} \rightarrow_{\mathtt{h}} \mathtt{I}(\mathtt{I}\,\mathtt{I}) \rightarrow_{\mathtt{h}} \mathtt{I}\,\mathtt{I} \rightarrow_{\mathtt{h}} \mathtt{I}$$

not simply typable

simply typable

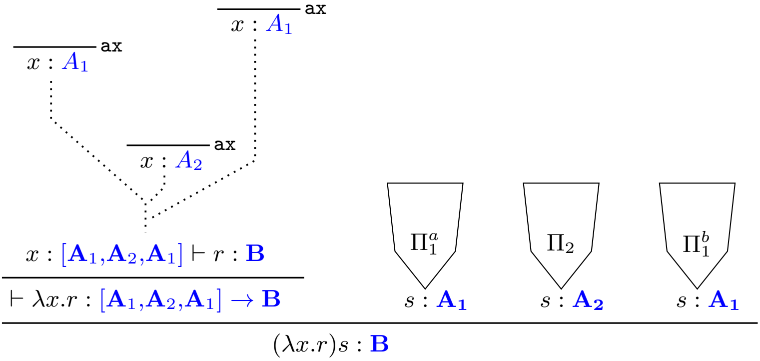
$$\boxed{F := [o]\rightarrow o}$$

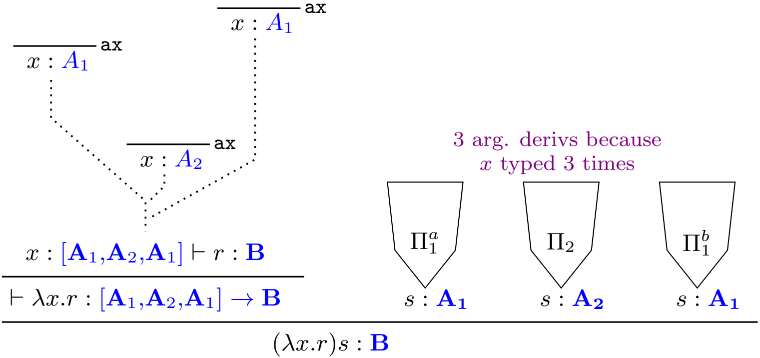
$$\dfrac{\mathtt{I} : F\rightarrow F \qquad \dfrac{\mathtt{I} : F\rightarrow F \qquad \mathtt{I} : F}{\mathtt{I}\,\mathtt{I} : F}}{\mathtt{I}(\mathtt{I}\,\mathtt{I}) : F}$$

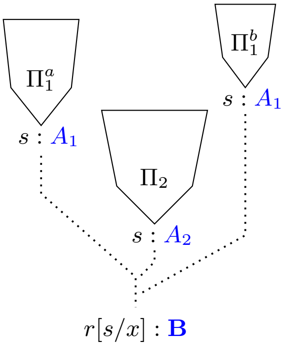
Typing $\mathtt{I}(\mathtt{I}\,\mathtt{I})$

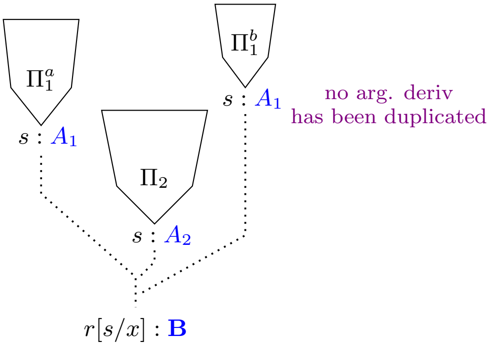
$$\dfrac{\dfrac{\dfrac{\dfrac{}{x : [F]\rightarrow F} \qquad \dfrac{\dfrac{}{x : [F]\rightarrow F} \qquad \dfrac{}{x : F}}{x\,x : F}}{x(x\,x) : F}}{\lambda x.x(x\,x) : [[F]\rightarrow F, [F]\rightarrow F, F]\rightarrow F} \qquad \dfrac{\dfrac{}{y : F}}{\mathtt{I} : [F]\rightarrow F} \qquad \dfrac{\dfrac{}{y : F}}{\mathtt{I} : [F]\rightarrow F} \qquad \dfrac{\dfrac{}{y : o}}{\mathtt{I} : F\ (=[o]\rightarrow o)}}{(\lambda x.x(x\,x))\mathtt{I} : F}$$

Why do deriv. decrease under eval.?

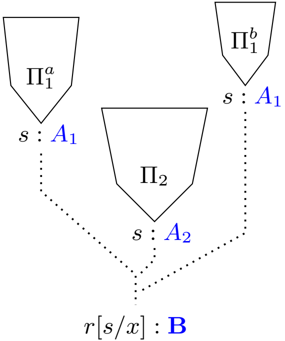



ax
$x : A_1$
ax
$x : A_1$
ax
$x : A_2$
$x : [\mathbf{A}_1, \mathbf{A}_2, \mathbf{A}_1] \vdash r : \mathbf{B}$
$\vdash \lambda x.r : [\mathbf{A}_1, \mathbf{A}_2, \mathbf{A}_1] \to \mathbf{B}$
3 arg. derivs because
x typed 3 times
Π_1^a
$s : \mathbf{A_1}$
Π_2
$s : \mathbf{A_2}$
Π_1^b
$s : \mathbf{A_1}$
$(\lambda x.r)s : \mathbf{B}$


Π_1^a
$s : A_1$
Π_2
$s : A_2$
Π_1^b
$s : A_1$
$r[s/x] : \mathbf{B}$

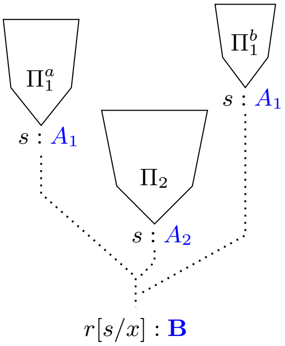

Π_1^a
$s : A_1$
Π_2
$s : A_2$
Π_1^b
$s : A_1$
no arg. deriv
has been duplicated
$r[s/x] : \mathbf{B}$

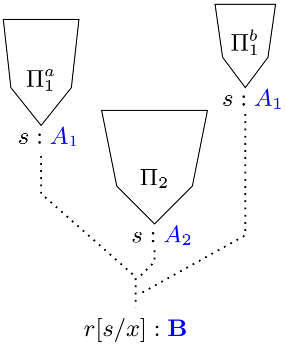
Non-idempotency, reduction and decrease



no arg. deriv
has been duplicated

$\rightsquigarrow$ the deriv. has
decreased in size


Π_1^a
$s : A_1$
Π_2
$s : A_2$
Π_1^b
$s : A_1$
$r[s/x] : \mathbf{B}$



Non-idempotency:
- **duplication** disallowed *(w.r.t. derivs)*
- derivations **decrease** in size

System $\mathscr{R}_0$ (Gardner 94-de Carvalho 07)

Types: $\tau,\ \sigma \quad ::= \quad o \quad | \quad [\sigma_i]_{i \in I} \rightarrow \tau$

- **intersection** = **multiset** of types $[\sigma_i]_{i \in I}$
- only on the left-h.s of $\rightarrow$ (*strictness*)

System $\mathscr{R}_0$ (Gardner 94-de Carvalho 07)

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$$\frac{}{x : [\tau] \vdash x : \tau}\ \texttt{ax} \qquad \frac{\Gamma; x : [\sigma_i]_{i \in I} \vdash t : \tau}{\Gamma \vdash \lambda x.t : [\sigma_i]_{i \in I} \to \tau}\ \texttt{abs}$$

$$\frac{\Gamma \vdash t : [\sigma_i]_{i \in I} \to \tau \qquad (\Gamma_i \vdash u : \sigma_i)_{i \in I}}{\Gamma +_{i \in I} \Gamma_i \vdash t\, u : \tau}\ \texttt{app}$$

Types: $\tau, \sigma \quad ::= \quad o \quad | \quad [\sigma_i]_{i \in I} \to \tau$

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Remark

- **Relevant** system (no weakening, *cf.* `ax`-rule)

System $\mathscr{R}_0$ (Gardner 94-de Carvalho 07)

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$$\frac{\Gamma \vdash t : [\sigma_i]_{i\in I} \to \tau \qquad (\Gamma_i \vdash u : \sigma_i)_{i\in I}}{\Gamma +_{i\in I} \Gamma_i \vdash t\,u : \tau}\ \texttt{app}$$

Remark

- **Relevant** system (no weakening, *cf.* `ax`-rule)
- **Non-idempotency ($\sigma \wedge \sigma \neq \sigma$):**
  in `app`-rule, pointwise multiset sum *e.g.*,

$$(x : [\sigma]; y : [\tau]) + (x : [\sigma, \tau]) = x : [\sigma, \sigma, \tau]; y : [\tau]$$

System $\mathscr{R}_0$ (Gardner 94-de Carvalho 07)

Types: $\quad \tau,\ \sigma \quad ::= \quad o \quad | \quad [\sigma_i]_{i\in I} \to \tau$

- **intersection** = **multiset** of types $[\sigma_i]_{i\in I}$
- only on the left-h.s of $\to$ (*strictness*)

$$\frac{}{x : [\tau] \vdash x : \tau}\ \texttt{ax} \qquad \frac{\Gamma; x : [\sigma_i]_{i\in I} \vdash t : \tau}{\Gamma \vdash \lambda x.t : [\sigma_i]_{i\in I} \to \tau}\ \texttt{abs}$$

$$\frac{\Gamma \vdash t : [\sigma_i]_{i\in I} \to \tau \qquad (\Gamma_i \vdash u : \sigma_i)_{i\in I}}{\Gamma +_{i\in I} \Gamma_i \vdash t\,u : \tau}\ \texttt{app}$$

$$\text{Types:} \qquad \tau,\ \sigma \quad ::= \quad o \quad | \quad [\sigma_i]_{i\in I} \to \tau$$

- **intersection** = **multiset** of types $[\sigma_i]_{i\in I}$
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$$\frac{}{x : [\tau] \vdash x : \tau}\ \texttt{ax} \qquad \frac{\Gamma; x : [\sigma_i]_{i\in I} \vdash t : \tau}{\Gamma \vdash \lambda x.t : [\sigma_i]_{i\in I} \to \tau}\ \texttt{abs}$$

$$\frac{\Gamma \vdash t : [\sigma_i]_{i\in I} \to \tau \qquad (\Gamma_i \vdash u : \sigma_i)_{i\in I}}{\Gamma +_{i\in I} \Gamma_i \vdash t\,u : \tau}\ \texttt{app}$$

Example (arg. typed n times):

$$\frac{\overline{x : [\sigma]\to\tau} \qquad \overline{y : \sigma}}{x\,y : \tau} \qquad \frac{\overline{x : [\,] \to \tau}}{x\,y : \tau} \qquad \frac{\overline{x : [\sigma, \tau, \sigma] \to \tau} \qquad \overline{y : \sigma} \qquad \overline{y : \tau} \qquad \overline{y : \sigma}}{x\,y : \tau}$$

$$\text{Types:} \qquad \tau,\ \sigma \quad ::= \quad o \quad | \quad [\sigma_i]_{i\in I} \to \tau$$

- **intersection** = **multiset** of types $[\sigma_i]_{i\in I}$
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$$\frac{}{x : [\tau] \vdash x : \tau}\ \mathtt{ax} \qquad \frac{\Gamma; x : [\sigma_i]_{i\in I} \vdash t : \tau}{\Gamma \vdash \lambda x.t : [\sigma_i]_{i\in I} \to \tau}\ \mathtt{abs}$$

$$\frac{\Gamma \vdash t : [\sigma_i]_{i\in I} \to \tau \qquad (\Gamma_i \vdash u : \sigma_i)_{i\in I}}{\Gamma +_{i\in I} \Gamma_i \vdash t\,u : \tau}\ \mathtt{app}$$

Example (arg. typed n times):

$$\frac{\dfrac{}{x : [\sigma]\to\tau} \qquad \dfrac{}{y : \sigma}}{x\,y : \tau} \qquad \frac{\dfrac{}{x : [\,] \to \tau}}{x\,y : \tau} \qquad \frac{\dfrac{}{x : [\sigma,\tau,\sigma] \to \tau} \qquad \dfrac{}{y : \sigma} \qquad \dfrac{}{y : \tau} \qquad \dfrac{}{y : \sigma}}{x\,y : \tau}$$

singleton domain
$\rightsquigarrow$ y typed 1 time

empty domain
$\rightsquigarrow$ y untyped

#domain = 3
$\rightsquigarrow$ y typed 3 times

System $\mathscr{R}_0$ (Gardner 94-de Carvalho 07)

$$\text{Types:} \quad \tau,\ \sigma \quad ::= \quad o \quad | \quad [\sigma_i]_{i\in I} \to \tau$$

- **intersection** = **multiset** of types $[\sigma_i]_{i\in I}$
- only on the left-h.s of $\to$ (*strictness*)

$$\frac{}{x : [\tau] \vdash x : \tau}\ \texttt{ax} \qquad \frac{\Gamma; x : [\sigma_i]_{i\in I} \vdash t : \tau}{\Gamma \vdash \lambda x.t : [\sigma_i]_{i\in I} \to \tau}\ \texttt{abs}$$

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Head redexes always typed!

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but an arg. may be typed 0 time

Properties of $\mathscr{R}_0$ (non-idempotent intersection)

Dynamics

- Subject Reduction (SR)
  typing stable under reduction
- Subject Expansion (SE)
  typing stable under anti-reduction

Quantitative information

Head eval. decreases the **size** of derivations

· *size of* Π := *number of judg. in* Π

· *types and judg. are not duplicated!*

· *true whenever the redex is typed*

Theorem (de Carvalho)

t *is* $\mathscr{R}_0$*-typable*

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$\cdot\, t \to_{\mathrm{h}}^{\ell} \lambda x_1 \ldots x_p.x\, t_1 \ldots t_q$ *(length to HNF)*

$\cdot\, f = p + q + 1$ *(size of HNF)*

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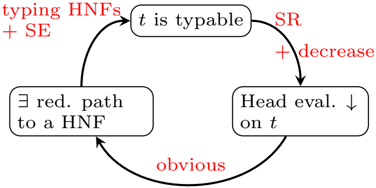

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Equality when Π **“minimal”** in some sense

Intersection: $\mathcal{I}, \mathcal{J} := [\mathcal{U}_k]_{k \in K}$

$\mathcal{U}, \mathcal{V} =: \langle \sigma_k \rangle_{k \in K}$: **Union**

Non-idempotent typing for $\lambda\mu$

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Features and properties

Syntax-direction, relevance, multiplicative rules, **accumulation of typing information**.

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app-rule based upon the *admissible* rule of ND:

$$\frac{A_1 \to B_1 \vee \ldots \vee A_k \to B_k \qquad A_1 \wedge \ldots \wedge A_k}{B_1 \vee \ldots \vee B_k}$$

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μ-red. may ***increase*** the number of nodes!

because of app. rule creation

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Take into account...

- multiplicity of unions ($\rightsquigarrow$ no increase)
- **arities** of saved functions ($\rightsquigarrow$ **decrease**)

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FSCD'17 (Kesner-V.)

Quantitative characterization of HN and SN in $\lambda\mu$

Plan

1. The Lambda-Mu Calculus
2. Non-idempotent intersection types
3. Capturing exact measures (length + normal form)

Toward exact typing

- **Persistent** elements
(remain in the NF)
- **Consuming** elements
(used during eval.)

ex: $(\lambda x.y@x@x)@z \to_\beta y@z@z$

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Inductive hypothesis
u_1 and u_2 typed with $\bullet$

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- Works for **leftmost-outermost** eval. (computes the full NF)

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(\textbf{Types}) & \sigma, \tau & ::= & \bullet \mid [\sigma_i]_{i \in I} \to \tau \mid [\sigma_i]_{i \in I} \twoheadrightarrow \tau \\
(\textbf{Exact types}) & \mathtt{Ex} & ::= & \bullet \mid \boxed{[\,\bullet\,] \twoheadrightarrow \mathtt{Ex}}
\end{array}
$$

$\Gamma \vdash t : \tau$ exact when τ exact & only exact types in Γ

- Works for **leftmost-outermost** eval. (computes the full NF)
- For **head eval.**, head args must be untyped

$$
\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{}{x : [\,\bullet\,] \twoheadrightarrow [\,\bullet\,] \twoheadrightarrow \bullet}\,\mathtt{ax} \quad u_1 : \bullet}{x\, u_1 : [\,\bullet\,] \twoheadrightarrow \bullet}\,\mathtt{app} \quad u_2 : \bullet}{x\, u_1\, u_2 : \bullet}\,\mathtt{app}}{\lambda x.x\, u_1\, u_2 : \bullet}}{\lambda yx.x\, u_1\, u_2 : \bullet}}{}
$$

- **Persistent** elements (remain in the NF)
- **Consuming** elements (used during eval.)

ex: $(\lambda x.y@x@x)@z \to_\beta y@z@z$

- Explicit **persistent** arrow: $\nrightarrow$ (new type constructor)
- One type constant: $\bullet$ (meaning "not applied")
- Help define **exact types** designed to capture **exact measures**

(**Types**)	σ, τ	$::=$	$\bullet \mid [\sigma_i]_{i \in I} \to \tau \mid [\sigma_i]_{i \in I} \nrightarrow \tau$
(**Exact types**)	$\mathtt{Ex}$	$::=$	$\bullet \mid \boxed{}$

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$$\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{}{x : [\,\bullet\,] \nrightarrow [\,\bullet\,] \nrightarrow \bullet}\,\texttt{ax} \qquad u_1 : \bullet}{x\,u_1 : [\,\bullet\,] \nrightarrow \bullet}\,\texttt{app} \qquad u_2 : \bullet}{x\,u_1\,u_2 : \bullet}\,\texttt{app}}{\lambda x.x\,u_1\,u_2 : \bullet}}{\lambda yx.x\,u_1\,u_2 : \bullet}$$

- Works for **leftmost-outermost** eval. (computes the full NF)
- For **head eval.**, head args must be untyped $\rightsquigarrow$ exact types must be redefined

Toward exact typing

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  $[\bullet] \twoheadrightarrow \mathtt{Ex} \rightsquigarrow [\ ] \twoheadrightarrow \mathtt{Ex}$

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Toward exact typing

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ex: $(\lambda x.y@x@x)@z \to_\beta y@z@z$

- Explicit **persistent** arrow: $\not\to$ (new type constructor)
- One type constant: $\bullet$ (meaning "not applied")
- Help define **exact types** designed to capture **exact measures**

$$
\begin{array}{lll}
(\textbf{Types}) & \sigma,\tau & ::= \bullet \mid [\sigma_i]_{i\in I} \to \tau \mid [\sigma_i]_{i\in I} \not\to \tau \\
(\textbf{Exact types}) & \mathtt{Ex} & ::= \bullet \mid [\ ] \not\to \mathtt{Ex}
\end{array}
$$

$\Gamma \vdash t : \tau$ exact when τ exact & only exact types in Γ

$$
\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{}{x : [\ ] \not\to [\ ] \not\to \bullet}\ \mathtt{ax}}{x\, u_1 : [\ ] \not\to \bullet}\ \mathtt{app}}{x\, u_1\, u_2 : \bullet}\ \mathtt{app}}{\lambda x.x\, u_1\, u_2 : \bullet}}{\lambda yx.x\, u_1\, u_2 : \bullet}
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$\Gamma \vdash t : \tau$ exact when τ exact & only exact types in Γ

$$\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{}{x : [\ \ ] \not\to [\ \ ] \not\to \bullet}\ \mathtt{ax}}{x\,u_1 : [\ \ ] \not\to \bullet}\ \mathtt{app}}{x\,u_1\,u_2 : \bullet}\ \mathtt{app}}{\lambda x.x\,u_1\,u_2 : \bullet}}{\lambda yx.x\,u_1\,u_2 : \bullet}$$

$\mathtt{dom}([\ \ ] \not\to \mathtt{Ex}) = [\ \ ] \neq [\bullet],[\bullet,\bullet],\ldots$
$\rightsquigarrow$ head args. are *not typed*

u_1 and u_2 not typed anymore

$$\Gamma \vdash^{(\ell, f)} t : \tau$$

number of β-steps

size of the norm. form

Idea:
counting $\rightarrow$ gives ℓ
counting $\not\rightarrow$ gives f

$$\frac{}{x : [\sigma] \vdash^{(0,1)} x : \sigma} \; (\texttt{ax})$$

$$\frac{\Gamma, x : [\sigma_i]_{i \in I} \vdash^{(\ell, f)} t : \sigma}{\Gamma \vdash^{(\ell+1, f-\#I)} \lambda x.t : [\sigma_i]_{i \in I} \to \sigma} \; (\texttt{abs}) \qquad \frac{\Gamma, x : [\sigma_i]_{i \in I} \vdash^{(\ell, f)} t : \texttt{Ex} \qquad [\sigma_i]_{i \in I} \text{ exact}}{\Gamma \vdash^{(\ell, f+1)} \lambda x.t : \bullet} \; (\bullet_S)$$

$$\frac{\Gamma \vdash^{(\ell_t, f_t)} t : \mathcal{F} \qquad \Delta \Vdash^{(\ell_u, f_u)} u : \texttt{dom}(\mathcal{F})}{\Gamma + \Delta \vdash^{(\ell_t + \ell_u, f_t + f_u + \#_{\texttt{p}} \mathcal{F})} t\, u : \texttt{codom}(\mathcal{F})} \; (\texttt{app}_S) \qquad \frac{(\Gamma_k \vdash^{(\ell_k, f_k)} t : \tau_k)_{k \in K}}{+_k \Gamma_k \Vdash^{(+_k \ell_k, +_k f_k)} t : [\tau_k]_{k \in K}} \; \wedge$$

with $\mathcal{F}$ arrow, $\#_{\texttt{p}} ? \to ? = 0$ and $\#_{\texttt{p}} ? \not\rightarrow ? = 1$.

Systems $\mathcal{X}^{\lambda}_{\texttt{hd/lo}}$

$$\frac{}{x : [\sigma] \vdash^{(0,1)} x : \sigma}\ (\mathtt{ax})$$

$$\frac{\Gamma, x : [\sigma_i]_{i \in I} \vdash^{(\ell,f)} t : \sigma}{\Gamma \vdash^{(\ell+1, f-\#I)} \lambda x.t : [\sigma_i]_{i\in I} \to \sigma}\ (\mathtt{abs}) \qquad \frac{\Gamma, x : [\sigma_i]_{i \in I} \vdash^{(\ell,f)} t : \mathtt{Ex} \qquad [\sigma_i]_{i\in I} \text{ exact}}{\Gamma \vdash^{(\ell, f+1)} \lambda x.t : \bullet}\ (\bullet_S)$$

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Systems $\mathcal{X}^{\lambda}_{\mathtt{hd/lo}}$

Focus on. . .

Variables

$$\frac{}{x : [\sigma] \vdash^{(0,1)} x : \sigma}\ (\mathtt{ax})$$

$$\frac{\Gamma, x : [\sigma_i]_{i\in I} \vdash^{(\ell,f)} t : \sigma}{\Gamma \vdash^{(\ell+1, f-\#I)} \lambda x.t : [\sigma_i]_{i\in I} \to \sigma}\ (\mathtt{abs}) \qquad \frac{\Gamma, x : [\sigma_i]_{i\in I} \vdash^{(\ell,f)} t : \mathtt{Ex} \qquad [\sigma_i]_{i\in I} \text{ exact}}{\Gamma \vdash^{(\ell, f+1)} \lambda x.t : \bullet}\ (\bullet_S)$$

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with $\mathcal{F}$ arrow, $\#_{\mathtt{p}} ? \to ? = 0$ and $\#_{\mathtt{p}} ? \not\to ? = 1$.

Systems $\mathcal{X}^{\lambda}_{\mathtt{hd/lo}}$

Focus on. . .

$$\frac{}{x : [\sigma] \vdash^{(0,1)} x : \sigma} \text{(ax)}$$

$\cdot\, x$ is a n.f. $\rightsquigarrow \ell = 0$

$\cdot\, x$ is of size $\rightsquigarrow f = 1$

$$\frac{}{x : [\sigma] \vdash^{(0,1)} x : \sigma} \text{(ax)}$$

$$\frac{\Gamma, x : [\sigma_i]_{i \in I} \vdash^{(\ell,f)} t : \sigma}{\Gamma \vdash^{(\ell+1, f-\#I)} \lambda x.t : [\sigma_i]_{i\in I} \to \sigma} \text{(abs)} \qquad \frac{\Gamma, x : [\sigma_i]_{i \in I} \vdash^{(\ell,f)} t : \mathtt{Ex} \qquad [\sigma_i]_{i\in I} \text{ exact}}{\Gamma \vdash^{(\ell, f+1)} \lambda x.t : \bullet} (\bullet_S)$$

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with $\mathcal{F}$ arrow, $\#_{\mathtt{p}}? \to ? = 0$ and $\#_{\mathtt{p}}? \not\to ? = 1$.

Systems $\mathcal{X}^{\lambda}_{\mathtt{hd/lo}}$

Focus on...

$$\frac{}{x : [\sigma] \vdash^{(0,1)} x : \sigma}\ (\mathtt{ax})$$

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with $\mathcal{F}$ arrow, $\#_{\mathtt{p}}? \to ? = 0$ and $\#_{\mathtt{p}}? \not\to ? = 1$.

Systems $\mathcal{X}^{\lambda}_{\mathtt{hd/lo}}$

Focus on...

Consuming abstractions

used to type the abs. of redexes (initial or created)

e.g., $\mathtt{I}$ and $\lambda x.y$ in $(\mathtt{I}(\lambda x.y))\Delta$ (n.f $= y$)

$$\frac{}{x : [\sigma] \vdash^{(0,1)} x : \sigma} (\mathtt{ax})$$

$$\frac{\Gamma, x : [\sigma_i]_{i \in I} \vdash^{(\ell,f)} t : \sigma}{\Gamma \vdash^{(\ell+1, f - \#I)} \lambda x.t : [\sigma_i]_{i \in I} \to \sigma} (\mathtt{abs}) \qquad \frac{\Gamma, x : [\sigma_i]_{i \in I} \vdash^{(\ell,f)} t : \mathtt{Ex} \qquad [\sigma_i]_{i \in I} \text{ exact}}{\Gamma \vdash^{(\ell, f+1)} \lambda x.t : \bullet} (\bullet_S)$$

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with $\mathcal{F}$ arrow, $\#_{\mathtt{p}} ? \to ? = 0$ and $\#_{\mathtt{p}} ? \not\to ? = 1$.

Systems $\mathcal{X}^{\lambda}_{\mathtt{hd/lo}}$

Focus on...

Consuming abstractions

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e.g., I and $\lambda x.y$ in $(\mathtt{I}(\lambda x.y))\Delta$

(n.f $= y$)

$$\frac{\Gamma, x : [\sigma_i]_{i\in I} \vdash^{(\ell,f)} t : \sigma}{\Gamma \vdash^{(\ell+1, f-\#I)} \lambda x.t : [\sigma_i]_{i\in I} \to \sigma}\ (\mathtt{abs})$$

· $\lambda x.t$ contributes to 1 step $\rightsquigarrow \ell \leftarrow \ell + 1$

· all the occ. of x will be subst. $\rightsquigarrow f \leftarrow f - |I|$

$$\frac{}{x : [\sigma] \vdash^{(0,1)} x : \sigma}\ (\mathtt{ax})$$

$$\frac{\Gamma, x : [\sigma_i]_{i\in I} \vdash^{(\ell,f)} t : \sigma}{\Gamma \vdash^{(\ell+1, f-\#I)} \lambda x.t : [\sigma_i]_{i\in I} \to \sigma}\ (\mathtt{abs}) \qquad \frac{\Gamma, x : [\sigma_i]_{i\in I} \vdash^{(\ell,f)} t : \mathtt{Ex} \qquad [\sigma_i]_{i\in I} \text{ exact}}{\Gamma \vdash^{(\ell, f+1)} \lambda x.t : \bullet}\ (\bullet_S)$$

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with $\mathcal{F}$ arrow, $\#_{\mathtt{p}}? \to ? = 0$ and $\#_{\mathtt{p}}? \nrightarrow ? = 1$.

Systems $\mathcal{X}^{\lambda}_{\mathtt{hd/lo}}$

Focus on...

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$$\frac{\Gamma, x : [\sigma_i]_{i \in I} \vdash^{(\ell,f)} t : \sigma}{\Gamma \vdash^{(\ell+1, f-\#I)} \lambda x.t : [\sigma_i]_{i\in I} \to \sigma}\ (\mathtt{abs}) \qquad \frac{\Gamma, x : [\sigma_i]_{i \in I} \vdash^{(\ell,f)} t : \mathtt{Ex} \qquad [\sigma_i]_{i\in I} \text{ exact}}{\Gamma \vdash^{(\ell, f+1)} \lambda x.t : \bullet}\ (\bullet_S)$$

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with $\mathcal{F}$ arrow, $\#_{\mathtt{p}} ? \to ? = 0$ and $\#_{\mathtt{p}} ? \nrightarrow ? = 1$.

Systems $\mathcal{X}^{\lambda}_{\mathtt{hd/lo}}$

Focus on...

Persistent abstractions

used to type the unused abs.

e.g., $\lambda x.u$ in $\mathtt{I}(\lambda x.u)$

$$\frac{}{x : [\sigma] \vdash^{(0,1)} x : \sigma}\ (\mathtt{ax})$$

$$\frac{\Gamma, x : [\sigma_i]_{i \in I} \vdash^{(\ell,f)} t : \sigma}{\Gamma \vdash^{(\ell+1, f-\#I)} \lambda x.t : [\sigma_i]_{i \in I} \to \sigma}\ (\mathtt{abs}) \qquad \frac{\Gamma, x : [\sigma_i]_{i \in I} \vdash^{(\ell,f)} t : \mathtt{Ex} \qquad [\sigma_i]_{i \in I} \text{ exact}}{\Gamma \vdash^{(\ell, f+1)} \lambda x.t : \bullet}\ (\bullet_S)$$

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Systems $\mathcal{X}^{\lambda}_{\mathtt{hd/lo}}$

Persistent abstractions

used to type the unused abs.

e.g., $\lambda x.u$ in $\mathtt{I}(\lambda x.u)$

Focus on...

$$\frac{\Gamma, x : [\sigma_i]_{i\in I} \vdash^{(\ell,f)} t : \mathtt{Ex} \qquad [\sigma_i]_{i\in I} \text{ exact}}{\Gamma \vdash^{(\ell,f+1)} \lambda x.t : \bullet} (\bullet_S)$$

- $\lambda x.t$ not fired $\rightsquigarrow$ ℓ unchanged
- λx remains in n.f. $\rightsquigarrow$ $f \leftarrow f+1$
- "$[\sigma_i]_{i\in I}$ exact" depends on $\mathtt{hd}/\mathtt{lo}$

$$\frac{}{x : [\sigma] \vdash^{(0,1)} x : \sigma} (\mathtt{ax})$$

$$\frac{\Gamma, x : [\sigma_i]_{i\in I} \vdash^{(\ell,f)} t : \sigma}{\Gamma \vdash^{(\ell+1,f-\#I)} \lambda x.t : [\sigma_i]_{i\in I} \to \sigma} (\mathtt{abs}) \qquad \frac{\Gamma, x : [\sigma_i]_{i\in I} \vdash^{(\ell,f)} t : \mathtt{Ex} \qquad [\sigma_i]_{i\in I} \text{ exact}}{\Gamma \vdash^{(\ell,f+1)} \lambda x.t : \bullet} (\bullet_S)$$

$$\frac{\Gamma \vdash^{(\ell_t,f_t)} t : \mathcal{F} \qquad \Delta \Vdash^{(\ell_u,f_u)} u : \mathtt{dom}(\mathcal{F})}{\Gamma + \Delta \vdash^{(\ell_t+\ell_u, f_t+f_u+\#_{\mathtt{p}}\mathcal{F})} t\,u : \mathtt{codom}(\mathcal{F})} (\mathtt{app}_S) \qquad \frac{(\Gamma_k \vdash^{(\ell_k,f_k)} t : \tau_k)_{k\in K}}{+_k\Gamma_k \Vdash^{(+_k\ell_k, +_k f_k)} t : [\tau_k]_{k\in K}} \wedge$$

with $\mathcal{F}$ arrow, $\#_{\mathtt{p}}? \to ? = 0$ and $\#_{\mathtt{p}}? \nrightarrow ? = 1$.

Systems $\mathcal{X}^{\lambda}_{\mathtt{hd}/\mathtt{lo}}$

Focus on...

$$\frac{}{x : [\sigma] \vdash^{(0,1)} x : \sigma}\ (\texttt{ax})$$

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with $\mathcal{F}$ arrow, $\#_{\mathtt{p}}? \to ? = 0$ and $\#_{\mathtt{p}}? \nrightarrow ? = 1$.

Systems $\mathcal{X}^{\lambda}_{\texttt{hd/lo}}$

Focus on...

Applications

persistent or not

$$\frac{}{x : [\sigma] \vdash^{(0,1)} x : \sigma}\ (\mathtt{ax})$$

$$\frac{\Gamma, x : [\sigma_i]_{i \in I} \vdash^{(\ell,f)} t : \sigma}{\Gamma \vdash^{(\ell+1, f-\#I)} \lambda x.t : [\sigma_i]_{i \in I} \to \sigma}\ (\mathtt{abs}) \qquad \frac{\Gamma, x : [\sigma_i]_{i \in I} \vdash^{(\ell,f)} t : \mathtt{Ex} \qquad [\sigma_i]_{i \in I} \text{ exact}}{\Gamma \vdash^{(\ell,f+1)} \lambda x.t : \bullet}\ (\bullet_S)$$

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with $\mathcal{F}$ arrow, $\#_{\mathtt{p}}? \to ? = 0$ and $\#_{\mathtt{p}}? \nrightarrow ? = 1$.

Systems $\mathcal{X}^{\lambda}_{\mathtt{hd/lo}}$

Focus on...

Applications

persistent or not

$$\frac{\Gamma \vdash^{(\ell_t, f_t)} t : \mathcal{F} \qquad \Delta \Vdash^{(\ell_u, f_u)} u : \mathtt{dom}(\mathcal{F})}{\Gamma + \Delta \vdash^{(\ell_t + \ell_u, f_t + f_u + \#_{\mathtt{p}}\mathcal{F})} t\, u : \mathtt{codom}(\mathcal{F})}\ (\mathtt{app}_S)$$

· reds step counted in $\mathtt{abs}$

$\rightsquigarrow \ell \rightsquigarrow +\ 0$

· $f \rightsquigarrow +\#_{\mathtt{p}}\mathcal{F}$ ($:= \mathtt{pers?}1 : 0$)

$$\frac{}{x : [\sigma] \vdash^{(0,1)} x : \sigma}\ (\mathtt{ax})$$

$$\frac{\Gamma, x : [\sigma_i]_{i \in I} \vdash^{(\ell, f)} t : \sigma}{\Gamma \vdash^{(\ell+1, f - \# I)} \lambda x.t : [\sigma_i]_{i \in I} \to \sigma}\ (\mathtt{abs}) \qquad \frac{\Gamma, x : [\sigma_i]_{i \in I} \vdash^{(\ell, f)} t : \mathtt{Ex} \qquad [\sigma_i]_{i \in I} \text{ exact}}{\Gamma \vdash^{(\ell, f+1)} \lambda x.t : \bullet}\ (\bullet_S)$$

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with $\mathcal{F}$ arrow, $\#_{\mathtt{p}}? \to ? = 0$ and $\#_{\mathtt{p}}? \not\to ? = 1$.

Systems $\mathcal{X}^{\lambda}_{\mathtt{hd/lo}}$

Focus on...

Applications

persistent or not

$$\frac{\Gamma \vdash^{(\ell_t, f_t)} t : \mathcal{F} \qquad \Delta \Vdash^{(\ell_u, f_u)} u : \mathtt{dom}(\mathcal{F})}{\Gamma + \Delta \vdash^{(\ell_t+\ell_u, f_t+f_u+\#_{\mathtt{p}}\mathcal{F})} t\,u : \mathtt{codom}(\mathcal{F})}\ (\mathtt{app}_S)$$

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$\rightsquigarrow \ell \rightsquigarrow +\ 0$

· $f \rightsquigarrow +\#_{\mathtt{p}}\mathcal{F}$ (:= pers?1 : 0)

$$\frac{x : [\bullet] \not\rightarrow \bullet \qquad u : \bullet}{x\,u : \bullet}$$

$+\,\mathbf{1}$ (1 pers. @ created)

$$\frac{}{x : [\sigma] \vdash^{(0,1)} x : \sigma}\ (\mathtt{ax})$$

$$\frac{\Gamma, x : [\sigma_i]_{i\in I} \vdash^{(\ell, f)} t : \sigma}{\Gamma \vdash^{(\ell+1, f-\#I)} \lambda x.t : [\sigma_i]_{i\in I} \to \sigma}\ (\mathtt{abs}) \qquad \frac{\Gamma, x : [\sigma_i]_{i\in I} \vdash^{(\ell, f)} t : \mathtt{Ex} \qquad [\sigma_i]_{i\in I} \text{ exact}}{\Gamma \vdash^{(\ell, f+1)} \lambda x.t : \bullet}\ (\bullet_S)$$

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Systems $\mathcal{X}^{\lambda}_{\mathtt{hd/lo}}$

Focus on...

Applications

persistent or not

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Systems $\mathcal{X}^{\lambda}_{\mathtt{hd/lo}}$

Focus on...

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$$\frac{\lambda x.r : [\sigma_i]_{i \in I} \rightarrow \tau \qquad (s : \sigma_i)}{(\lambda x.r)s : \tau}$$

$+\,\mathbf{0}$ (no pers. @ created)

$$\frac{}{x : [\sigma] \vdash^{(0,1)} x : \sigma} (\mathtt{ax})$$

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Systems $\mathcal{X}^{\lambda}_{\mathtt{hd/lo}}$

$(\lambda x.x(x\,x))\mathtt{I}$ RELOADED

$$(\lambda x.x(x\,x))\mathtt{I} \to_{\mathtt{h}} \mathtt{I}(\mathtt{I}\,\mathtt{I}) \to_{\mathtt{h}} \mathtt{I}\,\mathtt{I} \to_{\mathtt{h}} \mathtt{I}$$

$\boxed{F := [o]\to o}$ used to type the persistent occ. of $\mathtt{I}$

- **blue:** persistent
- **red:** consuming

$$\dfrac{\mathtt{I} : [F]\to F \qquad \dfrac{\mathtt{I} : [F]\to F \qquad \mathtt{I} : F}{\mathtt{I}\,\mathtt{I} : F}}{\mathtt{I}(\mathtt{I}\,\mathtt{I}) : F}$$

Typing $\mathtt{I}(\mathtt{I}\,\mathtt{I})$

$$\dfrac{\dfrac{\dfrac{\dfrac{}{x : [F]\to F} \qquad \dfrac{\dfrac{}{x : [F]\to F} \qquad \dfrac{}{x : F}}{x\,x : F}}{x(x\,x) : F}}{\lambda x.x(x\,x) : [[F]\to F, [F]\to F, F]\to F} \qquad \dfrac{\dfrac{}{y : F}}{\mathtt{I} : [F]\to F} \qquad \dfrac{\dfrac{}{y : F}}{\mathtt{I} : [F]\to F} \qquad \dfrac{\dfrac{}{y : o}}{\mathtt{I} : F}}{(\lambda x.x(x\,x))\mathtt{I} : F}$$

$$(\lambda x.x(x\,x))\mathtt{I} \to_{\mathtt{h}} \mathtt{I}(\mathtt{I}\,\mathtt{I}) \to_{\mathtt{h}} \mathtt{I}\,\mathtt{I} \to_{\mathtt{h}} \mathtt{I}$$

$$\boxed{\begin{array}{l} F := [o]\to o \\ \qquad \rightsquigarrow F := \bullet \end{array}}$$

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Typing $\mathtt{I}(\mathtt{I}\,\mathtt{I})$

$$\dfrac{\dfrac{\dfrac{\dfrac{}{x : [F]\to F} \qquad \dfrac{\dfrac{}{x : [F]\to F} \qquad \dfrac{}{x : F}}{x\,x : F}}{x(x\,x) : F}}{\lambda x.x(x\,x) : [[F]\to F, [F]\to F, F]\to F} \qquad \dfrac{\dfrac{}{y : F}}{\mathtt{I} : [F]\to F} \qquad \dfrac{\dfrac{}{y : F}}{\mathtt{I} : [F]\to F} \qquad \dfrac{\dfrac{}{y : o}}{\mathtt{I} : F}}{(\lambda x.x(x\,x))\mathtt{I} : F}$$

$(\lambda x.x(x\,x))\texttt{I}$ RELOADED

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$$\dfrac{\dfrac{\dfrac{\dfrac{}{x : [\bullet]\to\bullet} \qquad \dfrac{\dfrac{}{x : [\bullet]\to\bullet} \qquad \dfrac{}{x : \bullet}}{x\,x : \bullet}}{x(x\,x) : \bullet}}{\lambda x.x(x\,x) : [[\bullet]\to\bullet, [\bullet]\to\bullet, \bullet]\to\bullet} \qquad \dfrac{\dfrac{}{y : \bullet}}{\texttt{I} : [\bullet]\to\bullet} \qquad \dfrac{\dfrac{}{y : \bullet}}{\texttt{I} : [\bullet]\to\bullet} \qquad \dfrac{\dfrac{}{y : \bullet}}{\texttt{I} : \bullet}}{(\lambda x.x(x\,x))\texttt{I} : \bullet}$$

$(\lambda x.x(x\,x))\mathtt{I}$ RELOADED

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Typing $\mathtt{I}(\mathtt{I}\,\mathtt{I})$

$$\dfrac{\dfrac{\dfrac{\dfrac{}{x : [\bullet]\to\bullet} \qquad \dfrac{\dfrac{}{x : [\bullet]\to\bullet} \qquad \dfrac{}{x : \bullet}}{x\,x : \bullet}}{x(x\,x) : \bullet}}{\lambda x.x(x\,x) : [[\bullet]\to\bullet, [\bullet]\to\bullet, \bullet]\to\bullet} \qquad \dfrac{\dfrac{}{y : \bullet}}{\mathtt{I} : [\bullet]\to\bullet} \qquad \dfrac{\dfrac{}{y : \bullet}}{\mathtt{I} : [\bullet]\to\bullet} \qquad \dfrac{\dfrac{}{y : \bullet}}{\mathtt{I} : \bullet}}{(\lambda x.x(x\,x))\mathtt{I} : \bullet}$$

We focus only on...

$$(\lambda x.x(x\,x))\mathtt{I} \to_{\mathtt{h}} \mathtt{I}(\mathtt{I}\,\mathtt{I}) \to_{\mathtt{h}} \mathtt{I}\,\mathtt{I} \to_{\mathtt{h}} \mathtt{I}$$

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Typing $\mathtt{I}(\mathtt{I}\,\mathtt{I})$

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used to type the persistent occ. of I

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$(\lambda x.x(x\,x))\mathtt{I}$ reloaded

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Typing $\mathtt{I}(\mathtt{I}\,\mathtt{I})$

$$\dfrac{\dfrac{\dfrac{\dfrac{}{x:[\bullet]\to\bullet} \qquad \dfrac{\dfrac{}{x:[\bullet]\to\bullet} \quad \dfrac{}{x:\bullet}}{x\,x:\bullet}}{x(x\,x):\bullet}}{\lambda x.x(x\,x):[[\bullet]\to\bullet,[\bullet]\to\bullet,\bullet]\to\bullet} \qquad \dfrac{\dfrac{}{y:\bullet}}{\mathtt{I}:[\bullet]\to\bullet} \qquad \dfrac{\dfrac{}{y:\bullet}}{\mathtt{I}:[\bullet]\to\bullet} \qquad \dfrac{\dfrac{}{y:\bullet}}{\mathtt{I}:\bullet}}{(\lambda x.x(x\,x))\mathtt{I}:\bullet}$$

consumingly typed (with rule ($\mathtt{abs}$))

$(\lambda x.x(x\,x))\mathtt{I}$ RELOADED

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$$\dfrac{\dfrac{\dfrac{\dfrac{}{x:[\bullet]\to\bullet} \qquad \dfrac{\dfrac{}{x:[\bullet]\to\bullet}\qquad\dfrac{}{x:\bullet}}{x\,x:\bullet}}{x(x\,x):\bullet}}{\lambda x.x(x\,x):[[\bullet]\to\bullet,[\bullet]\to\bullet,\bullet]\to\bullet} \qquad \dfrac{\dfrac{}{y:\bullet}\;(0,1)}{\mathtt{I}:[\bullet]\to\bullet}\;(1,0) \qquad \dfrac{\dfrac{}{y:\bullet}\;(0,1)}{\mathtt{I}:[\bullet]\to\bullet}\;(1,0) \qquad \dfrac{\dfrac{}{y:\bullet}}{\mathtt{I}:\bullet}}{(\lambda x.x(x\,x))\mathtt{I}:\bullet}$$

consumingly typed (with rule (abs))

persistently typed (with rule ($\bullet$))

$(\lambda x.x(x\,x))\texttt{I}$ RELOADED

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consumingly typed (with rule ($\texttt{abs}$))

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$$\dfrac{\dfrac{\dfrac{\dfrac{}{x:[\bullet]\to\bullet} \qquad \dfrac{\dfrac{}{x:[\bullet]\to\bullet} \qquad \dfrac{}{x:\bullet}}{x\,x:\bullet}}{x(x\,x):\bullet}}{\lambda x.x(x\,x):[[\bullet]\to\bullet,[\bullet]\to\bullet,\bullet]\to\bullet}\ (1,0) \qquad \dfrac{\dfrac{}{y:\bullet}\ (0,1)}{\mathtt{I}:[\bullet]\to\bullet}\ (1,0) \qquad \dfrac{\dfrac{}{y:\bullet}\ (0,1)}{\mathtt{I}:[\bullet]\to\bullet}\ (1,0) \qquad \dfrac{\dfrac{}{y:\bullet}\ (0,1)}{\mathtt{I}:\bullet}\ (0,2)}{(\lambda x.x(x\,x))\mathtt{I}:\bullet}\ (3,2)$$

consumingly typed (with rule ($\mathtt{abs}$))

persistently typed (with rule ($\bullet$))

Properties of $\mathcal{X}^{\lambda}_{\texttt{hd/lo}}$

Definition

- **Exact Intersection:** $[\sigma_i]_{i\in I}$ exact iff the σ_i are exact.
- **Exact judgment:** $\Gamma \vdash^{(\ell,f)} t : \texttt{Ex}$ with $\Gamma(x)$ exact for all x.
- **Exact derivation:** ccl with tight judg. (*local* criterion).

no need to look inside deriv.

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Theorem (H/WN)

Let $t \in \Lambda$. Then:

$\Gamma \vdash^{(\ell,f)} t : \tau$ exact iff

- $t \rightarrow^{\ell}_{\mathtt{hd/lo}} t'$ head/full n.f.
- $|t'|_{\mathtt{hd/lo}} = f$

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Theorem (H/WN)

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Theorem (SN)

Idem for SN and a **maximal** reduction strategy.

· *Just modify* $\texttt{dom}(\mathcal{F})$ *with* $\texttt{dom}_{\texttt{mx}}([\,] \to \tau) = [\bullet]$
· *Erasable args must now be typed*
· *Specify the size of what is erased in* t

Let $S \in \{\mathtt{hd}, \mathtt{lo}, \mathtt{mx}\}$ ($\rightsquigarrow$ S-exactness for inter. and union types)

$$\frac{}{x : [\mathcal{U}] \vdash^{(0,0,1)} x : \mathcal{U} \mid \emptyset}\ (\mathtt{ax}) \qquad \frac{\Gamma \vdash^{(\ell,m,f)} t : \mathcal{U} \mid \Delta}{\Gamma \vdash^{(\ell,m,f)} [\alpha]t \mid \Delta \vee \alpha : \mathcal{U}}\ (\mathtt{c}) \qquad (\wedge)$$

$$\frac{\Gamma \vdash^{(\ell,m,f)} t : \mathtt{UEx}_S \mid \Delta \qquad \Gamma(x)\ S-\text{exact}}{\Gamma \setminus\!\!\setminus x \vdash^{(\ell,m,f+1)} \lambda x.t : \langle \bullet \rangle \mid \Delta}\ (\bullet_S)$$

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Capturing exact measures in $\lambda\mu$ ($S = \mathtt{hd}, \mathtt{lo}, \mathtt{mx}$)

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type $\mathcal{U}$ stored in α *via* $\vee$

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See Sec. 3.3

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- $\mathcal{U}^{\uparrow}$ = transforms top-level $\not\rightarrow$ into $\rightarrow$ *(top-level = not nested in [_])*
  e.g., $([\langle \mathcal{I} \not\rightarrow \mathcal{V} \rangle] \not\rightarrow \langle [\overline{\bullet}] \not\rightarrow \overline{\bullet} \rangle)^{\uparrow} = [\langle \mathcal{I} \not\rightarrow \mathcal{V} \rangle] \rightarrow \langle [\overline{\bullet}] \rightarrow \overline{\bullet} \rangle$ with $\overline{\bullet} = \langle \bullet \rangle$.

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arity of $\mathcal{V}$

counts how many times $\mu.c$ is used:

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$\texttt{c}_{\texttt{p}}(\mathcal{U})$ counts how many persistent app. $\mu\alpha.c$ will create

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- $\mathtt{c_p}(\mathcal{U})$ = counts top-level $\not\rightarrow$ *(= number of future pers. @-nodes)*
  e.g., $\mathtt{c_p}(\langle \mathcal{I} \not\rightarrow \bar{\bullet}, \mathcal{I} \rightarrow \langle \mathcal{I} \not\rightarrow \langle \mathcal{J} \not\rightarrow \bar{\bullet} \rangle \rangle, \bar{\bullet} \rangle) = 3$

Parametrization

- Parametrized system (exact types + domains)

Exact types

(spec. normal forms)

- `hd`: empty domains
- `lo/mx`: singleton domains

Domains

(spec. if erasable args are typed)

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Theorem (Kesner,V)

let $S \in \{\mathtt{hd}, \mathtt{lo}, \mathtt{mx}\}$ and t a $\lambda\mu$-term. Then:

$t \to_S^{(\ell,m)} t'$ a S-NF with $|t'|_S = f$

iff $\Gamma \vdash^{(\ell,m,f)} t : \mathcal{U} \mid \Delta$ tight for some Γ, $\mathcal{U}$, Δ

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Bonus: completely factorized proofs!

Doggy bag

Non-idempotency:
forbid duplication of typing deriv.

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Simple proofs of termination.

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Sec. 3.2

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Sec. 3.2

Future work:
$\bar{\lambda}\mu\tilde{\mu}$ (seq. calc.)? Call-by-push-value?