Some applications of quantitative types inside and outside type theory

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Inria (LS2N CNRS)

July 8, 2018

Non-Idempotent

Intersection

Type Theory

Where we lie

Non-Idempotent

Intersection

Type Theory

Curry-Howard correspondence

Non-Idempotent

Intersection

characterizes:

- normalization
- complexity classes
- MSO-sat.

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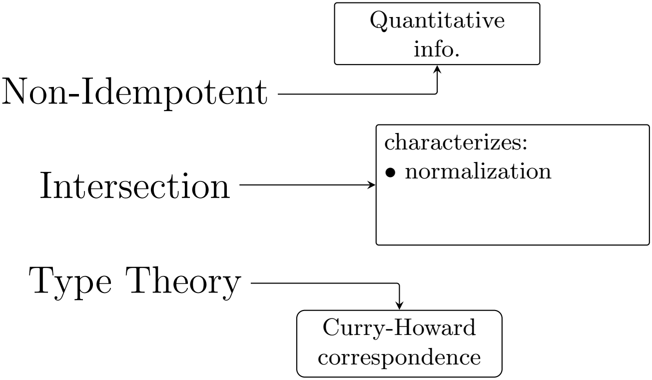
Where we lie



Quantitative info.
Non-Idempotent
characterizes:
• normalization
Intersection
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Plan

Intersection types (overview)

- Introduced by **Coppo-Dezani** (78-80) to "interpret more terms"
  - Charac. of Weak Norm. for λI-terms (no erasing β-step).
  - Extended later for λ-terms, head, weak or strong normalizatiion...
  - Filter models
- Model-checking
  - *Ong 06:* monadic second order (MSO) logic is decidable for higher-order recursion schemes (HORS)
  - *Kobayashi-Ong 09:* MSO is decidable for higher-order programs
    + using intersection types to simplify Ong's algorithm.
  - Refined by *Grellois-Melliès 14-15*
- Complexity:
  - Upper bounds for reduction sequences (*Gardner 94, de Carvalho 07*) or exact bounds (*Bernadet-Lengrand 11, Accattoli-Lengrand-Kesner, ICFP'18*).
  - *Terui 06:* upper bounds for terms in a red. sequence
  - *De Benedetti-Ronchi della Roccha 16*: characterization of FPTIME

Goal

Equivalences of the form

"the program t is typable iff it can reach a terminal state"

Idea: **several** certificates to a same subprogram (next slides).

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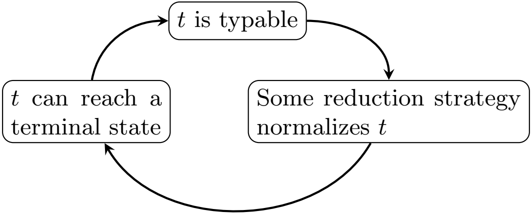

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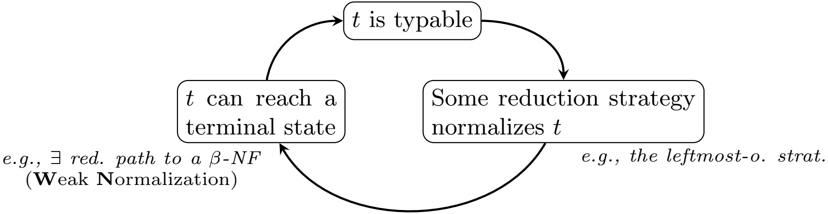

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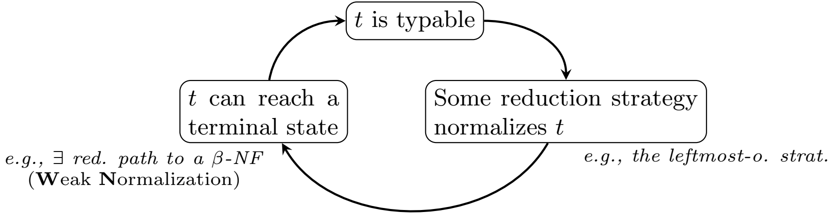

t is WN iff the leftmost-o. stategy terminates on t

nothing to do with types

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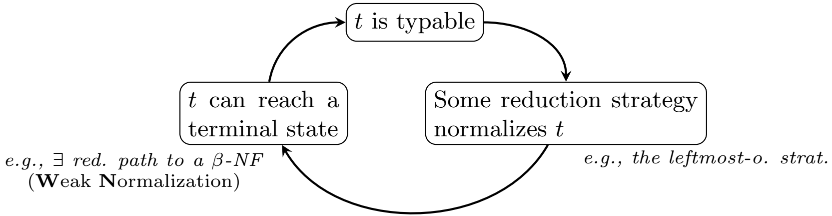

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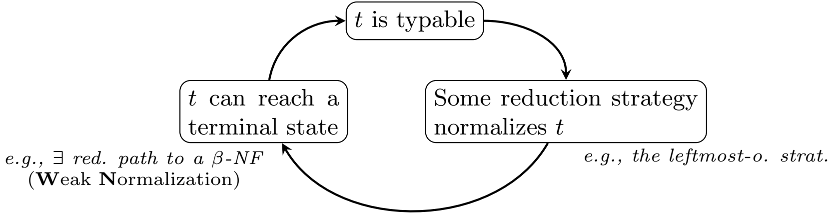

Intersection types

- Perhaps too expressive...
- ...but certify **reduction strategies!**

Intuitions (syntax)

- Naively, $A \wedge B$ stands for $A \cap B$:

t is of type $A \wedge B$ if t can be typed with A as well as B.

$$\frac{I : A \to A \qquad I : (A \to B) \to (A \to B)}{I : (A \to A) \wedge ((A \to B) \to (A \to B))} \wedge-\texttt{intro} \qquad \textit{(with } I = \lambda x.x\textit{)}$$

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- Intersection = kind of *finite polymorphism.*

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- But *less constrained*:

  assigning $x : o \wedge (o \rightarrow o') \wedge (o \rightarrow o) \rightarrow o$ is legal.

  *(**not** an instance of a polymorphic type except $\forall X.X :=$ `False`!)*

Subject Reduction and Subject Expansion

A good intersection type system should enjoy:

Subject Reduction (SR):
Typing is stable under reduction.

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Typing is stable under anti-reduction.

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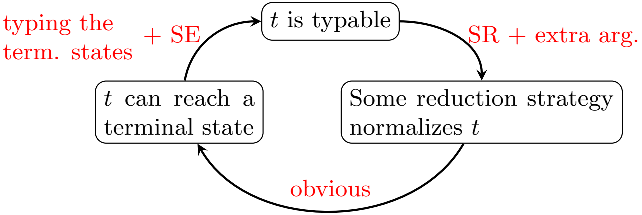

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$$\dfrac{\dfrac{x : A \vdash r : B}{\lambda x.r : A \to B}\,\texttt{abs} \qquad \dfrac{\Pi_s}{s : A}}{(\lambda x.r)s : B}\,\texttt{app}$$

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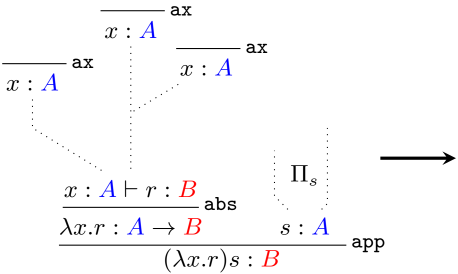

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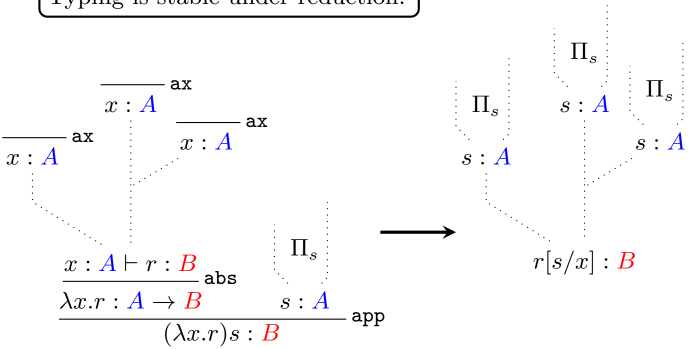

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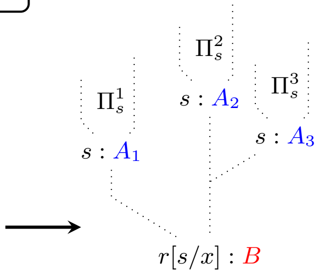

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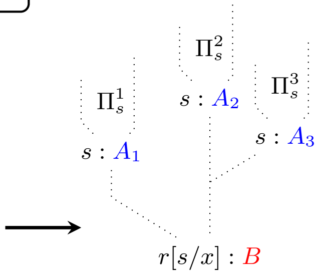

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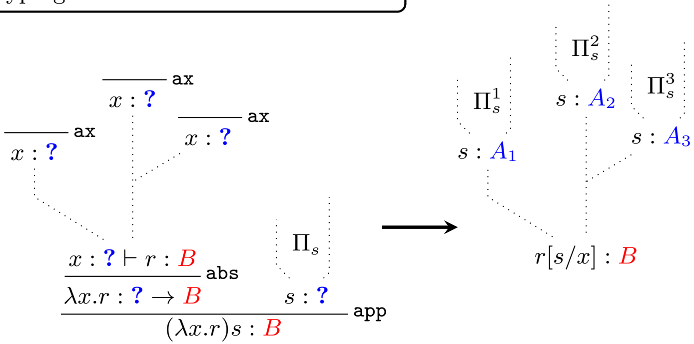

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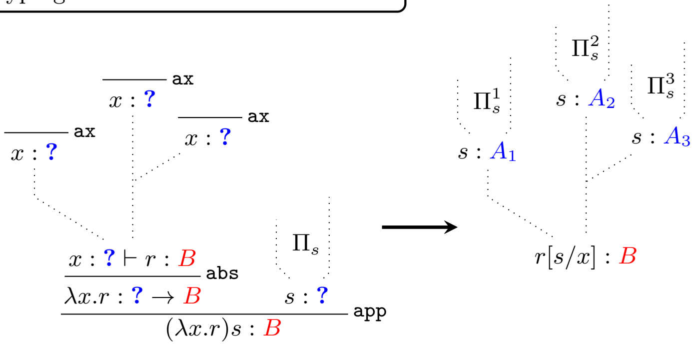

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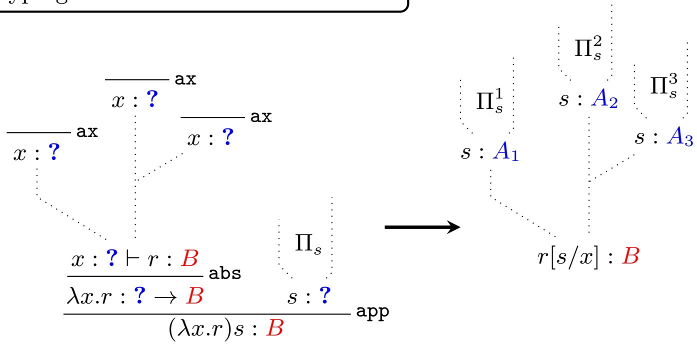

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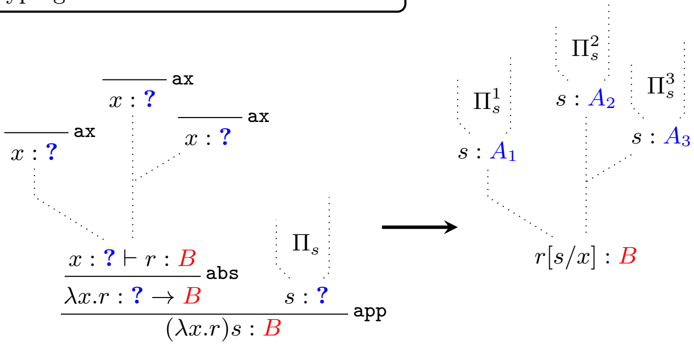

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$$x : A_1 \wedge A_2 \wedge A_3 \vdash x : A_i \;\; (i = 1, 2, 3)$$

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- Typing normal form: just structural induction (no clash).

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Disallow duplication for typing certificates.

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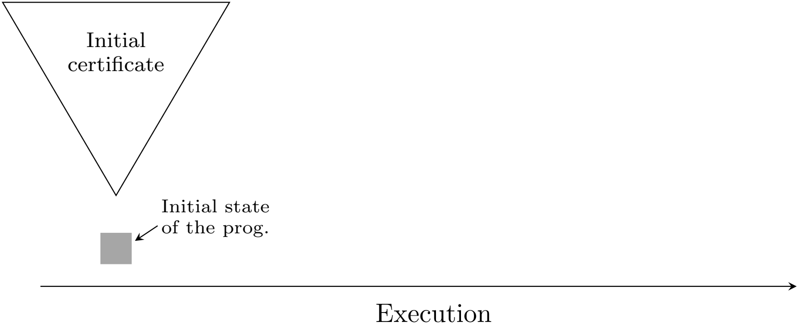

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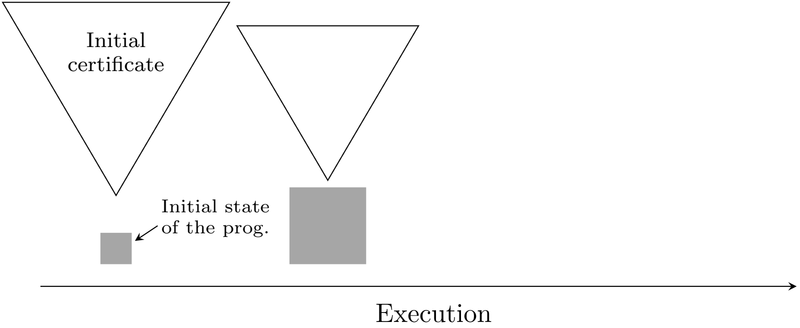

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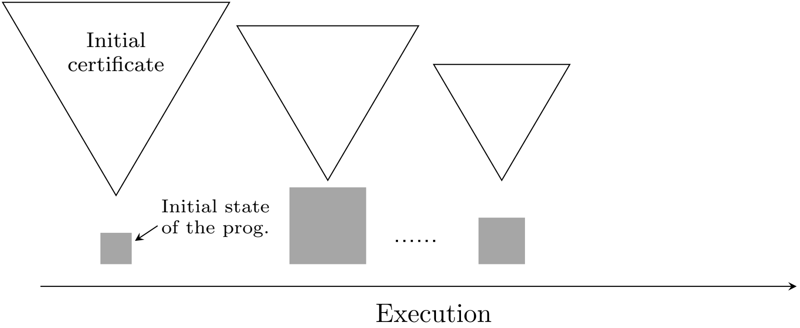

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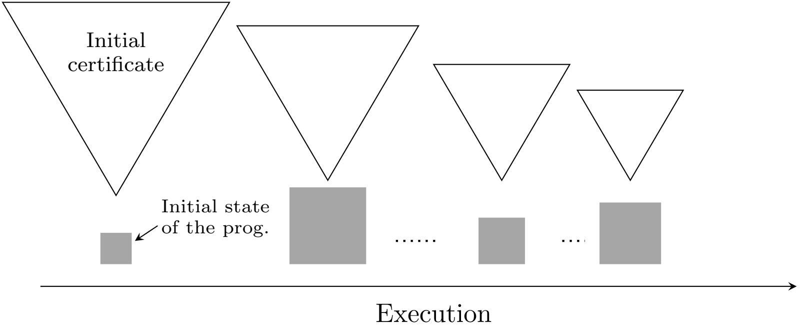

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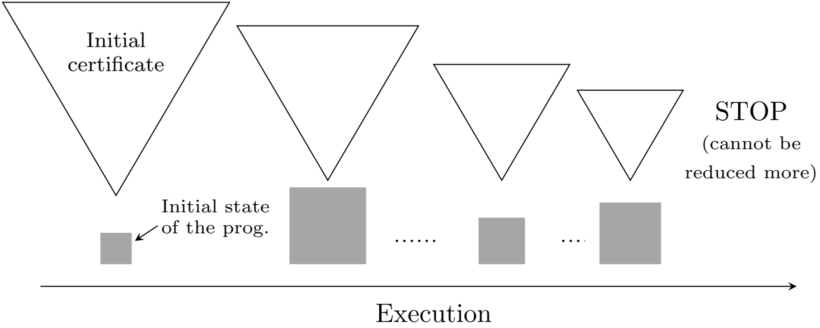

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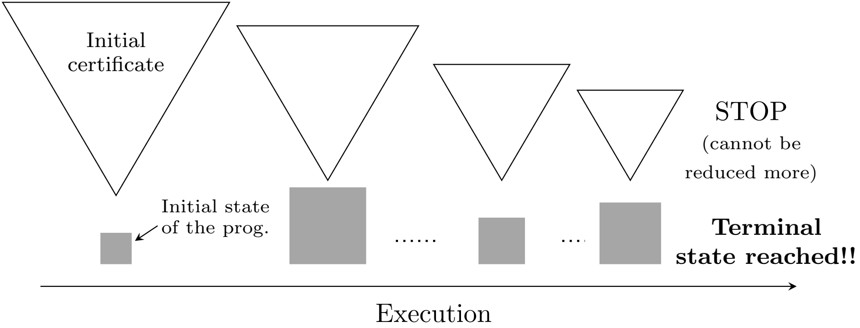

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Comparative (dis)advantages

- Insanely difficult to type a particular program.
- Whole type system **easier** to study!
  - Easier proofs of **termination**!
  - Easier proofs of **characterization**!
  - Easier to certify a **reduction strategy**!

Contents

The case of the λ-calculus

- Mechanics of non-idempotent intersection.
- Certification of reduction strategies. Quantitative intersection.
- Moving from various forms of normalization to others (head, weak, strong…)

$\lambda\mu$-calculus (classical logic)

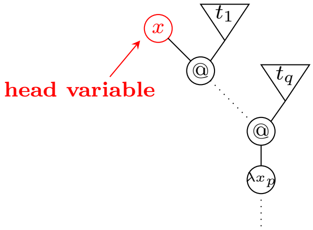
- Non-idempotent type theory adapts to more complicated operational semantics

Infinitary calculi

- Infinitary intersection type enables characterizing infinitary normalization (Klop's Problem).
- Dealing with unsoundness.
- Certification of an asymptotic reduction stategy.

Plan

Head Normalization (λ)



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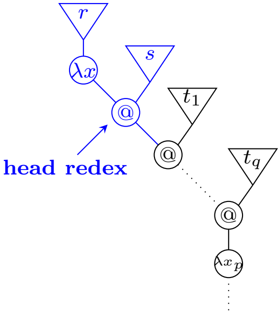

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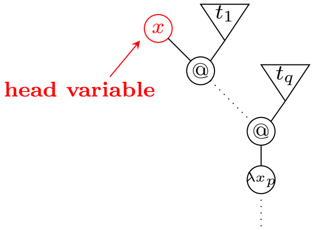

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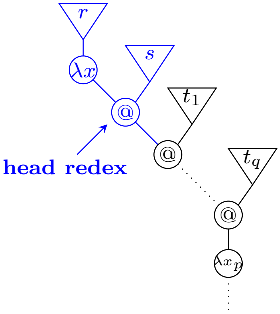

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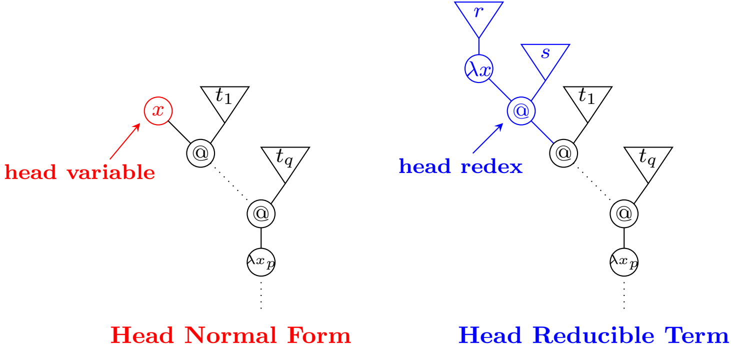

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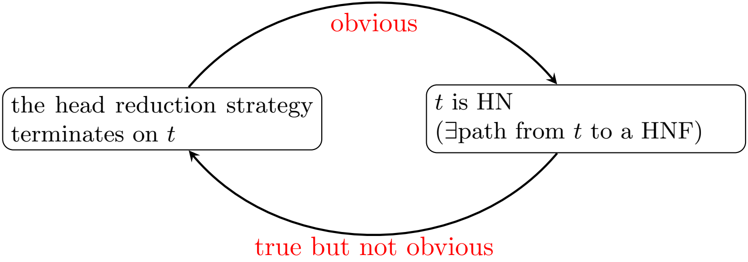

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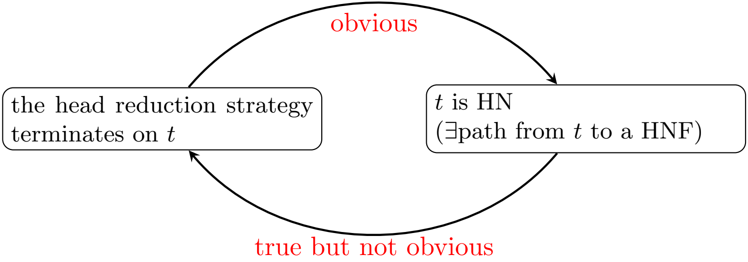

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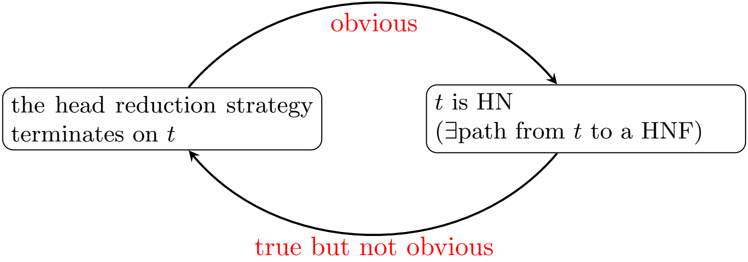

Intersection types come to help!

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- Collapsing $A \wedge B \wedge C$ into $[A, B, C]$ (**multiset**) $\rightsquigarrow$ no need for perm rules etc.

$$A \wedge B \wedge A := [A, B, A] = [A, A, B] \neq [A, B] \qquad [A, B, A] = [A, B] + [A]$$

System $\mathscr{R}_0$ (Gardner 94-de Carvalho 07)

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Remark

- **Relevant** system (no weakening, *cf.* `ax`-rule)

$$\text{Types:} \qquad \tau,\ \sigma \quad ::= \quad o \quad | \quad [\sigma_i]_{i\in I} \to \tau$$

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Remark

- **Relevant** system (no weakening, *cf.* `ax`-rule)
- **Non-idempotency ($\sigma \wedge \sigma \neq \sigma$):**
  in `app`-rule, pointwise multiset sum *e.g.*,

$$(x : [\sigma]; y : [\tau]) + (x : [\sigma, \tau]) = x : [\sigma, \sigma, \tau]; y : [\tau]$$

System $\mathscr{R}_0$ (Gardner 94-de Carvalho 07)

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Head redexes always typed!

Properties ($\mathscr{R}_0$)

- **Weighted Subject Reduction**
  - Reduction preserves types and environments, and...
  - ... *head* reduction strictly decreases the number of nodes of the deriv. tree (`size`).

    (actually, holds for any typed redex)

- **Subject Expansion**
  - Anti-reduction preserves types and environments.

Theorem (de Carvalho)

Let t be a λ-term. Then equivalence between:

1. *t is typable (in $\mathscr{R}_0$)*
2. *t is HN*
3. *the head reduction strategy terminates on t ($\rightsquigarrow$ **certification!**)*

Bonus (quantitative information)

If Π types t, then `size`(Π) bounds the number of **steps** of the head red. strategy on t

Head vs Weak and Strong Normalization

Let t be a λ-term.

- **Head normalization (HN):**
  there is a path from t to a head normal form.
- **Weak normalization (WN):**
  there is *at least one path* from t to a β**-Normal Form** (NF)
- **Strong normalization (SN):**
  there is *no infinite path* starting at t.

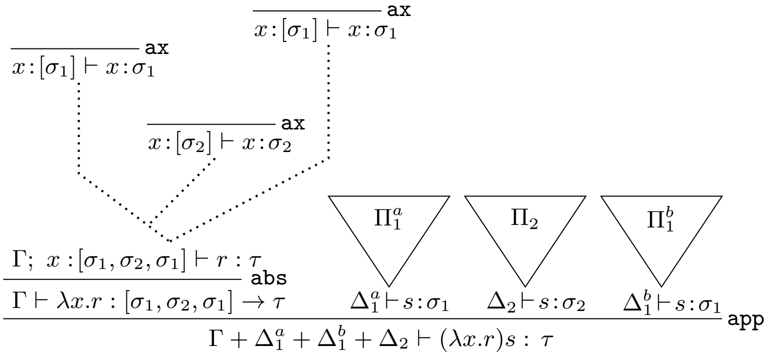
$$\boxed{\mathrm{SN} \Rightarrow \mathrm{WN} \Rightarrow \mathrm{HN}}$$

Nota Bene: $y\,\Omega$ HNF but not WN $\qquad$ $(\lambda x.y)\Omega$ WN but not SN

Characterizing Weak and Strong Normalization

HN	System $\mathscr{R}_0$ $\boxed{\textit{any} \text{ arg. can be left } \textit{untyped}}$	$\mathtt{sz}(\Pi)$ bounds the number of *head* reduction steps
WN	System $\mathscr{R}_0$ + **unforgetfulness criterion** $\boxed{\textit{non-erasable} \text{ args must be typed}}$	$\mathtt{sz}(\Pi)$ bounds the number of leftmost-outermost red. steps (and more)
SN	Modify system $\mathscr{R}_0$ with **choice operator** $\boxed{\textit{all} \text{ args must be typed}}$	$\mathtt{sz}(\Pi)$ bounds the length of *any* reduction path

From a typing of $(\lambda x.r)s$ … to a typing of $r[s/x]$



From a typing of $(\lambda x.r)s$ … to a typing of $r[s/x]$

$$\dfrac{}{x:[\sigma_1]\vdash x:\sigma_1}\,\texttt{ax} \qquad \dfrac{}{x:[\sigma_2]\vdash x:\sigma_2}\,\texttt{ax} \qquad \dfrac{}{x:[\sigma_1]\vdash x:\sigma_1}\,\texttt{ax}$$

$$\dfrac{\dfrac{\Gamma;\ x:[\sigma_1,\sigma_2,\sigma_1]\vdash r:\tau}{\Gamma\vdash\lambda x.r:{\color{red}[\sigma_1,\sigma_2,\sigma_1]\to\tau}}\,\texttt{abs} \qquad \begin{array}{c}\Pi_1^a\\ \vdots\\ \Delta_1^a\vdash s:\sigma_1\end{array} \qquad \begin{array}{c}\Pi_2\\ \vdots\\ \Delta_2\vdash s:\sigma_2\end{array} \qquad \begin{array}{c}\Pi_1^b\\ \vdots\\ \Delta_1^b\vdash s:\sigma_1\end{array}}{\Gamma+\Delta_1^a+\Delta_1^b+\Delta_2\vdash(\lambda x.r)s:\ \tau}\,\texttt{app}$$

Subject reduction and expansion in $\mathscr{R}_0$

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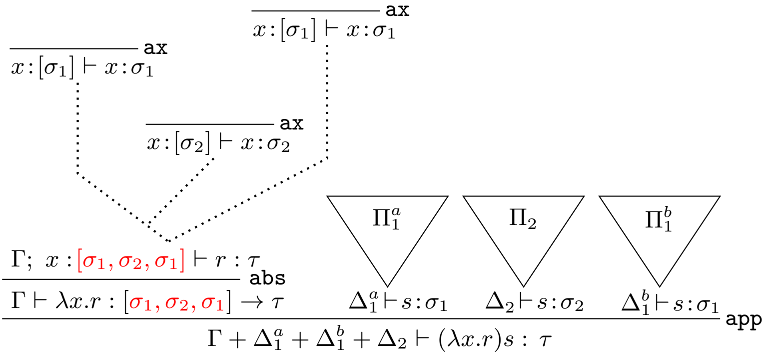

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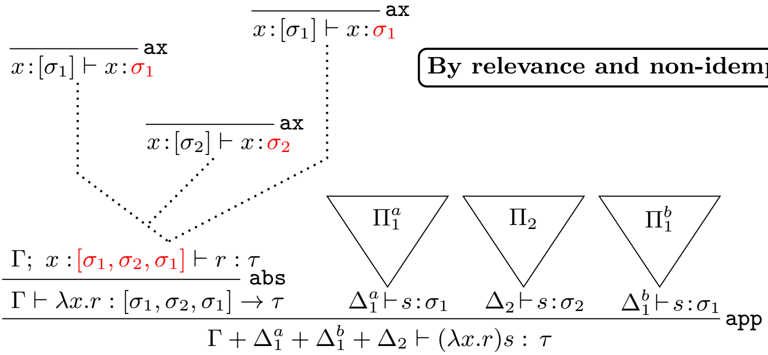

By relevance and non-idempotency!

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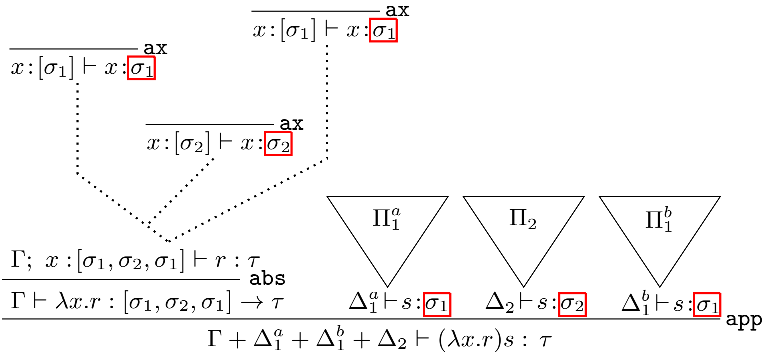

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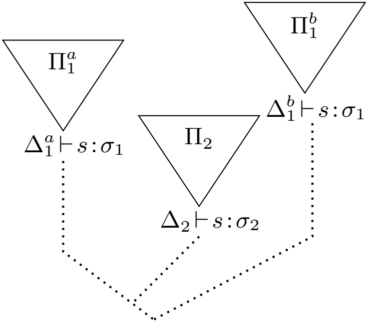

$$\Gamma + \Delta_1^a + \Delta_1^b + \Delta_2 \vdash r[s/x] : \tau$$

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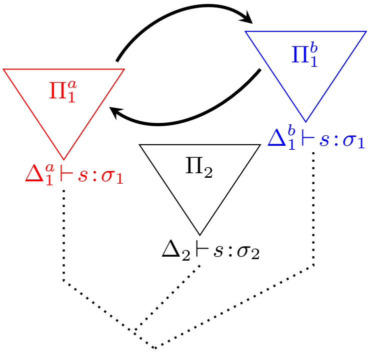

$\boxed{\textbf{Non-determinism of SR}}$

$\Gamma + \Delta_1^a + \Delta_1^b + \Delta_2 \vdash r[s/x] : \tau$

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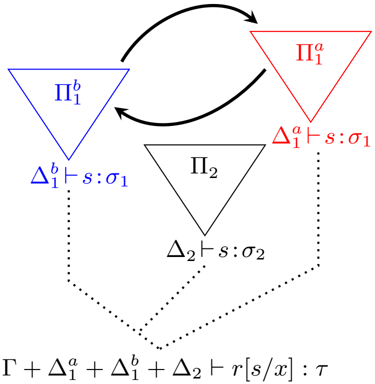

Non-determinism of SR

Plan

The Lambda-Mu Calculus

- Intuit. logic + Peirce's Law $((A \to B) \to A) \to A$ gives classical logic.

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$$\boxed{\textbf{classical logic}} \longleftrightarrow \boxed{\textbf{backtracking}}$$

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  $\leadsto$ the **Curry-Howard** iso extends to classical logic

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- **Parigot 92**: $\lambda\mu$-calculus = computational interpretation of classical *natural deduction* (*e.g., vs.* $\bar{\lambda}\mu\tilde{\mu}$).
  judg. of the form $A, A \to B \vdash A \mid B, C$

$$\dfrac{\dfrac{\dfrac{\dfrac{}{(A \to B) \to A \vdash (A \to B) \to A} \qquad \dfrac{\dfrac{}{A \vdash A, B}}{\vdash A \to B, A}}{(A \to B) \to A \vdash A, A}}{(A \to B) \to A \vdash A}}{\vdash ((A \to B) \to A) \to A}$$

Standard Style

$$
\dfrac{\dfrac{}{(A \to B) \to A \vdash (A \to B) \to A} \qquad \dfrac{\dfrac{}{A \vdash A, \color{red}{B}}}{\vdash A \to B, A}}{\dfrac{(A \to B) \to A \vdash A, A}{\dfrac{(A \to B) \to A \vdash A}{\vdash ((A \to B) \to A) \to A}}}
$$

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$$\dfrac{\dfrac{\dfrac{\dfrac{}{(A\to B)\to A\vdash (A\to B)\to A \mid} \qquad \dfrac{\dfrac{\dfrac{}{A\vdash A\mid B}}{A\vdash B\mid A}\,\texttt{act}}{\vdash A\to B\mid A}}{(A\to B)\to A\vdash A\mid A}}{(A\to B)\to A\vdash A\mid}}{\vdash ((A\to B)\to A)\to A\mid}$$

Focussed Style

In the right hand-side of $\Gamma \vdash F \mid \Delta$

- 1 active formula F
- inactive formulas Δ

$$
\begin{prooftree}
\AxiomC{}
\UnaryInfC{$(A \to B) \to A \vdash (A \to B) \to A \mid$}
\AxiomC{}
\UnaryInfC{$A \vdash A \mid B$}
\RightLabel{$\texttt{act}$}
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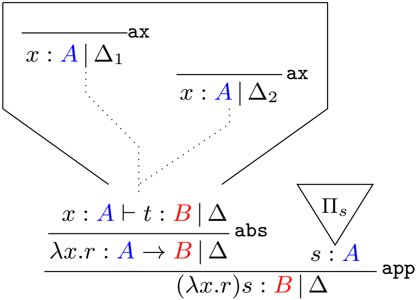
$\boxed{\text{How do we adapt the non-idempotent machinery to } \lambda\mu\text{?}}$

Cut-elimination steps (classical case)



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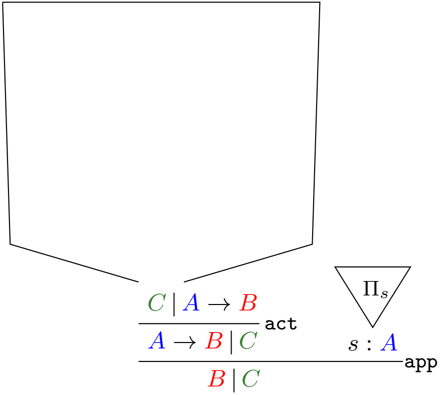
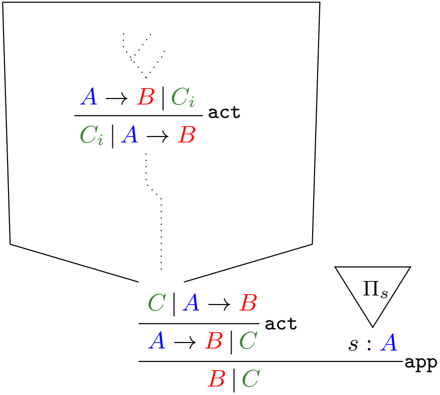
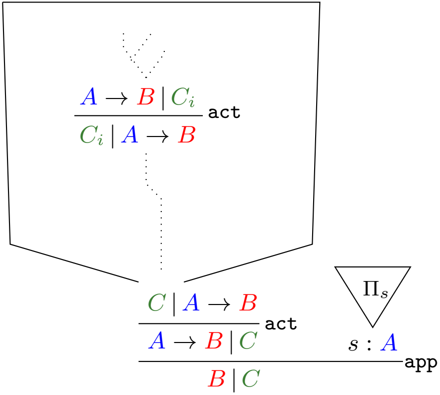



$A \to B \mid C_i$
act
$C_i \mid A \to B$
$C \mid A \to B$
act
$A \to B \mid C$
Π_s
$s : A$
app
$B \mid C$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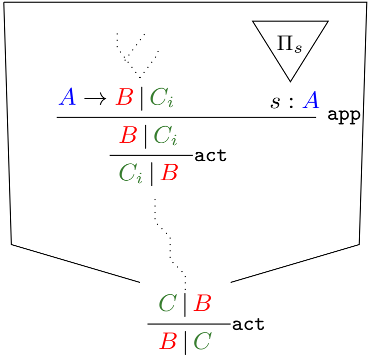
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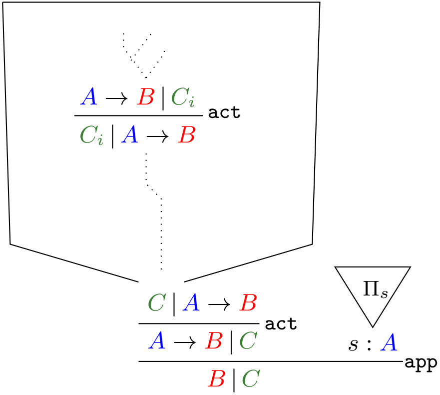

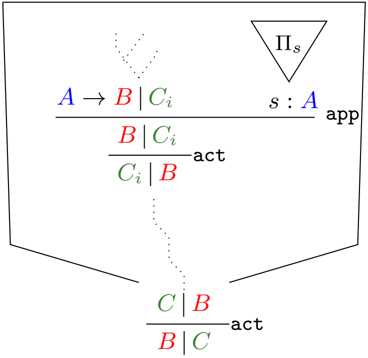

- Duplication of s
- Creation of app-rules
- B saved instead of $A \to B$

Intersection: $\mathcal{I}, \mathcal{J} := [\mathcal{U}_k]_{k \in K}$

$\mathcal{U}, \mathcal{V} =: \langle \sigma_k \rangle_{k \in K}$: **Union**

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The Typing System

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Syntax-direction, relevance, multiplicative rules, **accumulation of typing information**.

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$$\frac{A_1 \to B_1 \vee \ldots \vee A_k \to B_k \qquad A_1 \wedge \ldots \wedge A_k}{B_1 \vee \ldots \vee B_k} \qquad \left(vs. \frac{A \to B \quad A}{B} \right)$$

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$$\boxed{\texttt{call-cc} : [[[A] \to B] \to A] \to \langle A, A \rangle \qquad \text{vs.} \qquad ((A \to B) \to A) \to A}$$

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Syntax-direction, relevance, multiplicative rules, **accumulation of typing information**.

- **app**-rule based upon the *admissible* rule of ND:

$$\frac{A_1 \to B_1 \vee \ldots \vee A_k \to B_k \qquad A_1 \wedge \ldots \wedge A_k}{B_1 \vee \ldots \vee B_k} \qquad \left(vs. \frac{A \to B \quad A}{B} \right)$$

$$\boxed{\texttt{call--cc} : [[[A] \to B] \to A] \to \langle A, A \rangle \qquad \text{vs.} \qquad ((A \to B) \to A) \to A}$$

System $\mathcal{H}_{\lambda\mu}$ (Head Normalization)

- **Weighted Subject Reduction + Subject Expansion**

$$\texttt{size}(\Pi) = \begin{cases} \text{number of nodes of } \Pi \; + \\ \text{size of the } \textbf{type arities} \text{ of all the names of commands } + \\ \textbf{multiplicities} \text{ of arguments in all the } \textbf{app. nodes} \end{cases}$$

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Let t be a $\lambda\mu$-term. Equiv. between:

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- Small-step version.

Plan

Infinitary calculi

- Infinitary λ-trees provide various semantics to the λ-calculus.

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- Main idea:

Productive terms
- may not terminate…
- …but keep on outputting info. (*e.g.*, sub-HNF)
- *sound* infinite red. sequence

vs.

Meaningless terms
- do not output any info. ever (even a head variable)
- unsound infinite red. sequences

Productive vs. Unproductive Reduction

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Productive reduction: $\Delta_f := \lambda x.f(xx)$ $\quad \mathtt{Y}_f := \Delta_f \Delta_f$ "Curry f"

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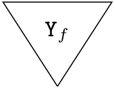

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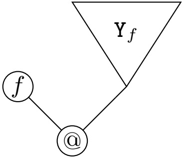

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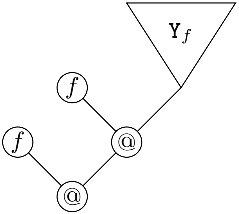

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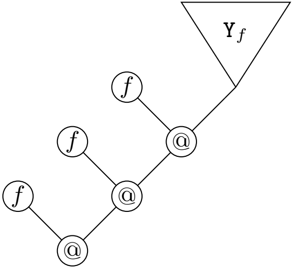

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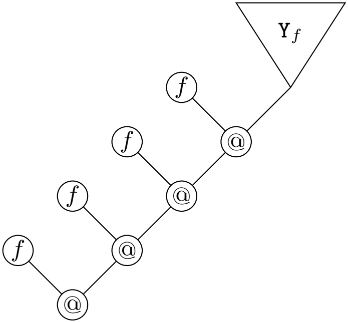

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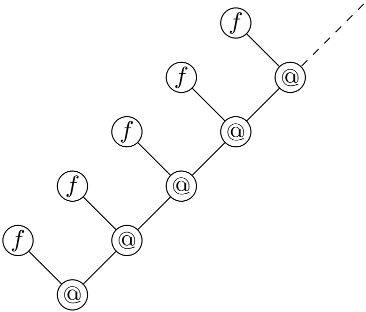

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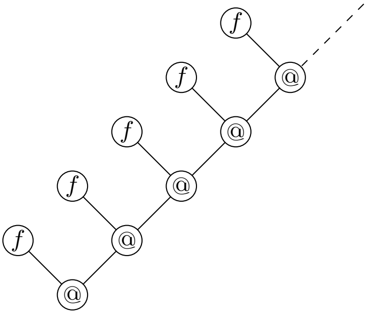

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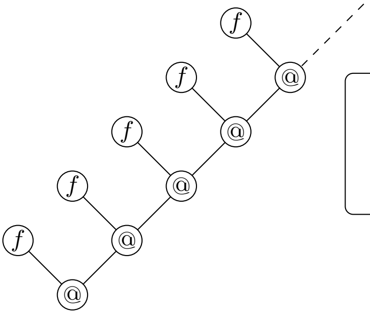

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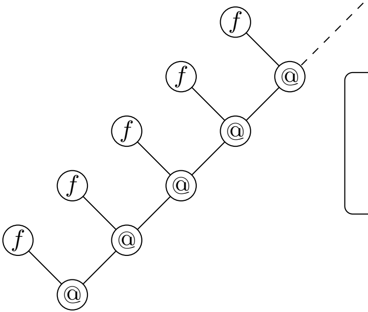

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Unproductive reduction: let $\Delta = \lambda x.x\,x$, $\Omega = \Delta\,\Delta$

$$\Omega \to \Omega \to \Omega \to \Omega \to \Omega \to \Omega \to \ldots$$

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Multiset intersection:

- ⊕ syntax-direction
- ⊖ non-determinism of proof red.
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  $[\sigma, \tau, \sigma] = [\underset{?}{\sigma}, \tau] + [\underset{?}{\sigma}]$.

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- Solution: **sequential** intersection

> **System $\mathtt{S}$**
> $\rightsquigarrow$ replace $[\sigma_i]_{i \in I}$ with $(k \cdot \sigma_k)_{k \in K}$

- **Tracking:** $(3 \cdot \sigma, 5 \cdot \tau, 9 \cdot \sigma) = (3 \cdot \sigma, 5 \cdot \tau) \uplus (9 \cdot \sigma)$

Characterization of infinitary WN

Proposition

In System $\mathtt{S}$:

- Validity (aka *approximability*) can be defined.
- *SR:* typing is stable by productive ∞-reduction.
- *SE: approximable* typing stable by productive ∞-expansion.

Theorem (V,LiCS'17)

- *A ∞-term t is ∞-WN iff t is unforgetfully typable by means of an approximable derivation* $\rightsquigarrow$ *Klop's Problem solved*
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Bonus: positive answer to TLCA Problem #20

System $\mathtt{S}$ also provides a type-theoretic characterization of the **hereditary permutations** (not possible in the inductive case, Tatsuta [LiCS'07]).

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Lemma: if $t \to_\infty t'$ HNF, then $t \to_{\mathtt{h}}^* t'_0$ HNF (finite sequence)

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Can *inductive* non-idem. inter. type systems help simplify proofs of infinitary confluence?

Plan

Intersection types *via* Grothendieck construction [Mazza,Pellissier,V, POPL2018]

- Categorical generalization of ITS *à la* Melliès-Zeilberger.
- Type systems = 2-operads (see below).

Type systems as 2-operads

- Level 1: $\Gamma \vdash t : B$ $\qquad t =$ *multimorphism* from Γ to B.
- Level 2: if $\Gamma \vdash t : B \overset{\text{SR}}{\rightsquigarrow} \Gamma \vdash t' : B$,
  $t \rightsquigarrow t' =$ *2-morphism* from t to t'.

- Construction of an ITS via a Grothendieck construction (pullbacks).
- **Modularity:** retrieving automatically
  e.g., Coppo-Dezani, Gardner, $\mathscr{R}_0$, call-by-value + $\mathcal{H}_{\lambda\mu}$ (use *cyclic* 2-operads)

The story continues...

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[Mazza,Pellissier,V, POPL2018]

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Damiano Mazza
Polyadic approximations and intersection types (ITRS/DCM joint invited talk)
*Sunday **4:30 pm**, Maths Seminar C5*

Luc Pellissier
Generalized generalized species of structure and resource modalities (Linearity/TLLA)
*Sunday **2 pm**, Blavatnik Seminar Room 1*

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Delia Kesner
Quantitative types: from Foundations to Applications (ITRS/DCM joint invited talk)
*Sunday **9 am**, Maths Seminar C5*

Thank you for your attention!

next talk in Floc

Every λ-term is meaningful in the infinitary relation model (Lics)
Monday 5:20 pm, *Math LT3*