The Expressive Power of Coinductive Rigid Types with non-Idempotent Intersection

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PLAN

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- STS: a variable x can be assigned only one type (that can be used several times).
- ITS: a variable can be typed several times, with different types. $x : A \wedge B \wedge B \wedge C$.
- *Example:* usually, xx cannot be typed in STS, but xx can be typed in ITS: if x is assigned $A \wedge (A \to B)$, then $xx : B$ is derivable.

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  In-between possibility: **rigidity** (paradigm: sequences)

CONTENTS OF THIS TALK

- We consider here two coinductive ITS, namely $\mathscr{M}$ and $\mathtt{S}$
  - In $\mathscr{M}$ (adapted from Gardner[94]/de Carvalho[07]), intersection is represented by means of multisets.
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- Forget about the order (inside a sequence):
  - A $\mathtt{S}$-type T **collapses** into a $\mathscr{M}$-type $\overline{T} = \tau$
  - A $\mathtt{S}$-context C collapses into a $\mathscr{M}$-context $\overline{C} = \Gamma$.
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  - Collapsing the judgments, a $\mathtt{S}$-derivation P collapses into a $\mathscr{M}$-derivation $\overline{P} = \Pi$.
- **Question 1 (full collapse?):** for all $\mathscr{M}$-derivation Π, is there a $\mathtt{S}$-derivation P that collapses into Π ? (easy for types and contexts)

PLAN

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- We cannot always perform the union of sequences.

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- *Example:* $[x, y, x]$ is the **collapse** of $(2 \cdot x, 3 \cdot y, 5 \cdot x)$.
- Equalities: $[x, y, x] = [x, x, y]$ but $(2 \cdot x, 3 \cdot y, 5 \cdot x) \neq (2 \cdot x, 3 \cdot x, 5 \cdot y)$ Equality is said to be **tight** for sequences (synctactic equality) and **loose** for multisets.

PLAN

STRICT INTERSECTION TYPES

- Let $\mathscr{X}$ be a countable set of type variables (metavariable α).
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- The application typing rule generally relies upon equality of intersection types (see next slide).

TYPING APPLICATION

- Symple types:

$$\frac{\Gamma \vdash t : \sigma \to \tau \qquad \Delta \vdash u : \sigma'}{\Gamma, \Delta \vdash t\, u : \tau}\ \mathtt{app}$$

Constraint: $\sigma = \sigma'$ (Γ and Δ do not type the same variables)

Typing Application

- Intersection types:

$$\dfrac{\Gamma \vdash t : \bigwedge_{i\in I} \sigma_i \to \tau \qquad (\Delta_i \vdash u : \sigma'_i)^{i \in I'}}{\Gamma \cup \bigcup_{i\in I} \Delta_i \vdash t\,u : \tau}\ \texttt{app}$$

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- Rigid ITS: i=sequence, $\wedge = \cup$ (disjoint): L = R is $(S_k)_{k\in K} = (S'_k)_{k\in K'}$
  ($C(x), D_k(x)$ disjoint for all x)

TYPING RULES OF $\mathscr{M}_0$ (GARDNER / DE CARVALHO)

Contexts (Γ, Δ): collection of $x : [\tau_i]_{i \in I}$.

$$\frac{}{x : [\tau] \vdash x : \tau}\ \mathrm{ax} \qquad \frac{\Gamma, x : [\sigma_i]_{i\in I} \vdash t : \tau}{\Gamma \vdash \lambda x.t : [\sigma_i]_{i\in I} \to \tau}\ \mathrm{abs}$$

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Remark

- Multiset sum: $[\alpha, \beta, \alpha] + [\alpha, \beta, \gamma, \gamma] = [\alpha, \alpha, \alpha, \beta, \beta, \gamma, \gamma]$
- No implicit contraction: accumulation of typing information .

EXAMPLE

Typing $\Delta = \lambda x.xx$ (with application arity = 3):

$$\dfrac{\dfrac{\dfrac{}{x : [[\alpha, \beta, \alpha] \to \alpha] \vdash x : [\alpha, \beta, \alpha] \to \alpha}\ \text{ax} \quad \dfrac{}{x : [\alpha] \vdash x : \alpha}\ \text{ax} \quad \dfrac{}{x : [\beta] \vdash x : \beta}\ \text{ax} \quad \dfrac{}{x : [\alpha] \vdash x : \alpha}\ \text{ax}}{x : [\alpha, \beta, \alpha, [\alpha, \beta, \alpha] \to \alpha] \vdash xx : \alpha}\ \text{ap}}{\vdash \lambda x.xx : [\alpha, \beta, \alpha, [\alpha, \beta, \alpha] \to \alpha] \to \alpha}\ \text{abs}$$

SUBJECT REDUCTION PROPERTY FOR $\mathscr{M}_0$

If $\Pi \triangleright \Gamma \vdash t : \tau$ and $t \to t'$, then $\exists \Pi' \triangleright \Gamma \vdash t' : \tau$

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$$(\lambda x.r)s \to r[s/x]$$

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Axiom leaves
typing x inside Π_r

$$\begin{array}{c}\Pi_r \\ \vdots \quad \left(\dfrac{}{x : [\sigma_i] \vdash x : \sigma_i}\,\texttt{ax}\right)_{i\in I} \\ \vdots\end{array}$$

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Vocabulary:
We say each **association** (between x-axiom leaves and arg-derivations) or **reduction choice**, yields a **reduced derivation** Π' typing $r[s/x]$.

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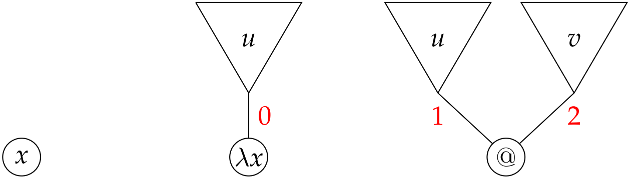
Observation:
If a type σ occurs several times in $[\sigma_i]_{i \in I}$, there can be several associations, each one yielding a possibly different reduced derivation Π'.

PLAN

∞-TERMS

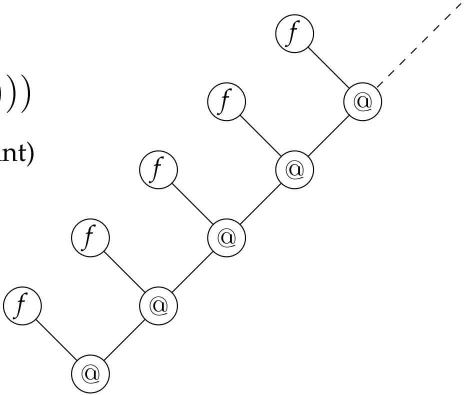


Variable x **Abstraction** $\lambda x.u$ **Application** $u\,v$

001-TERMS

$$f^\omega := f(f(f(\ldots)))$$

i.e. $f^\omega = f(f^\omega)$ (fixpoint)



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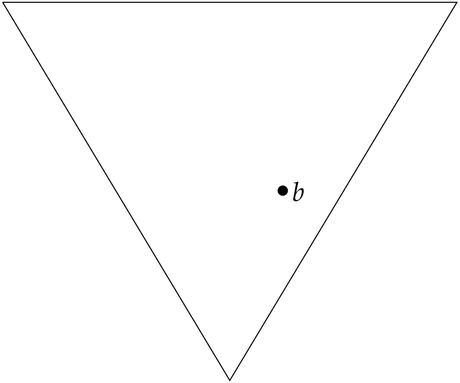

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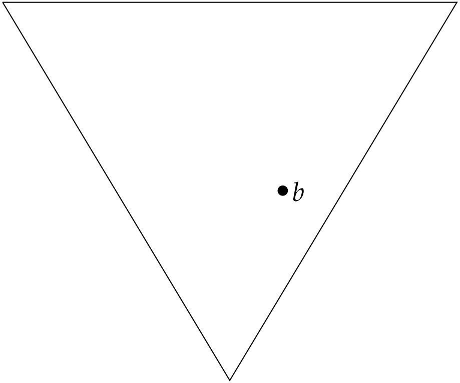

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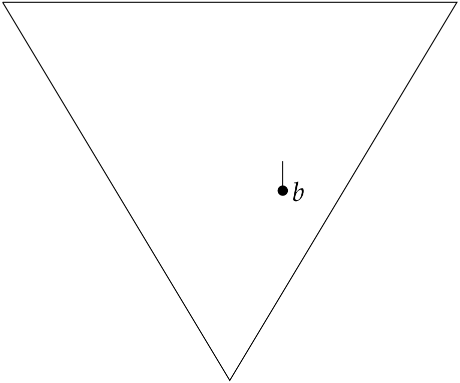
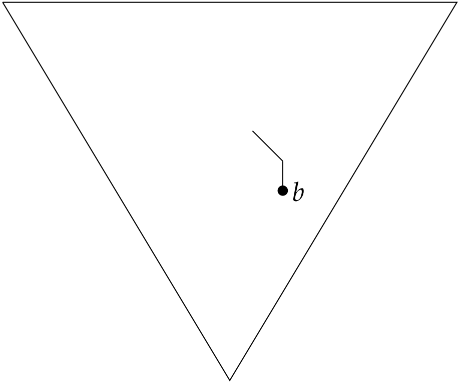
001-terms



- Start from $b \in \mathrm{supp}(t)$
- Move $\uparrow$ or $\nwarrow$

001-TERMS



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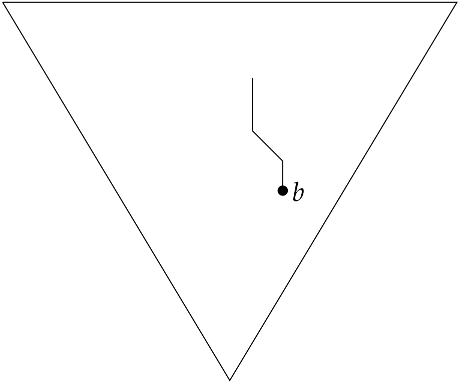

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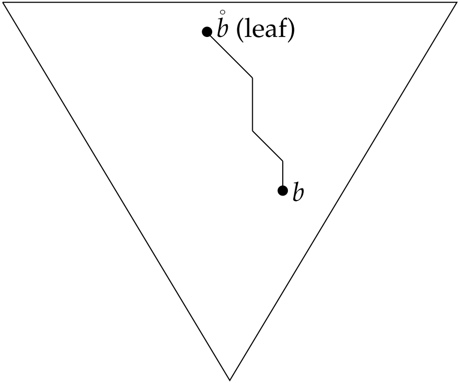

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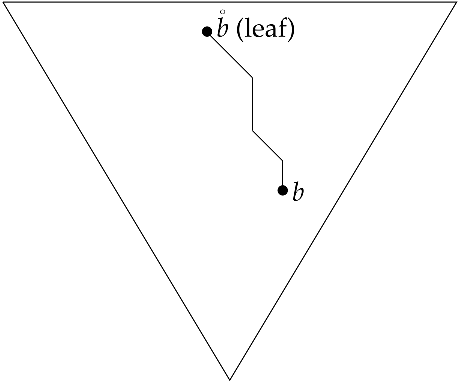

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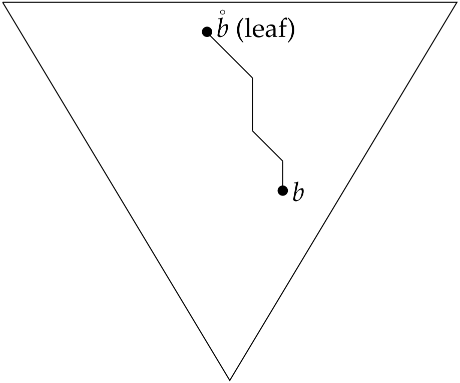
Λ^{001}: the set of ∞-terms t s.t.:

$$b \text{ is an infinite branch of } t \Rightarrow \texttt{ad}(b) = \infty.$$



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- Notion of full **type** (resp. **sequence type**) **isomorphism** when $T \equiv T'$ (resp. $(S_k)_{k\in K} \equiv (S'_k)_{k\in K'}$).

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The set Deriv of rigid derivations is *coinductively* generated by:

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- Application constraint 2: the contexts must be disjoint, so that no **track conflict** occurs.

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- **Question 2:** loss of expressivity compared to multiset intersection systems ?
- Every symbol is identified (notion of **biposition**): possibility of trace a type through typing rules, of residual of a type after subj. red.

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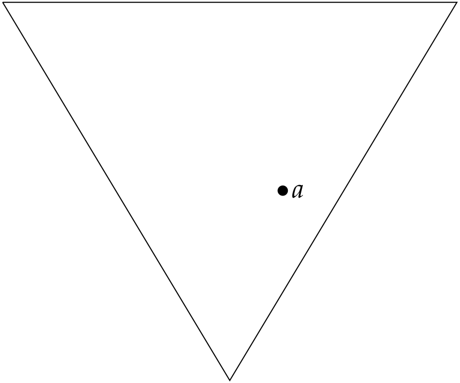
- Use of multisets: cannot distinguish two occ. of the same type in a multiset, trace a type inside a derivation, define residuals of a type after sub. red.

Plan

Global Typing Constraints

- P is a tree, $A := \mathrm{supp}(P)$ and $P(a) = C(a) \vdash t|_a : T(a)$ for all a.



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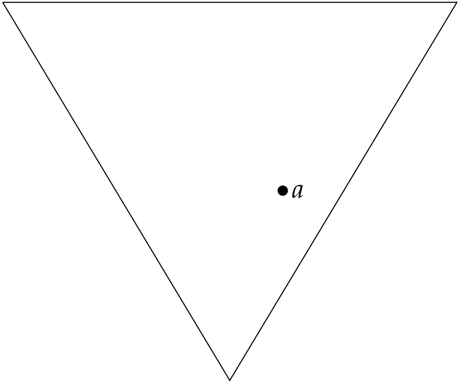

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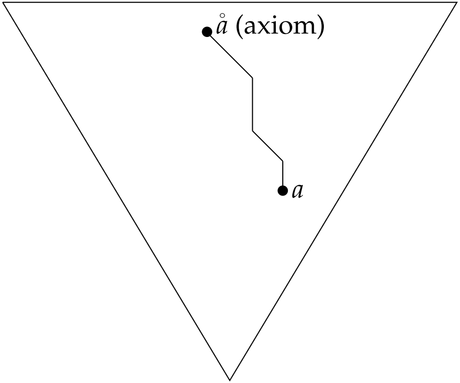

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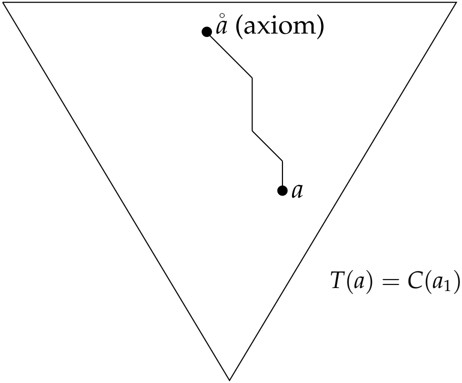

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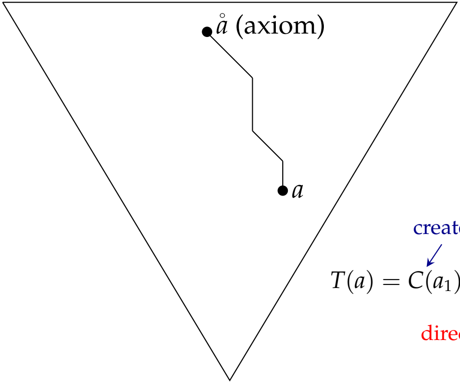

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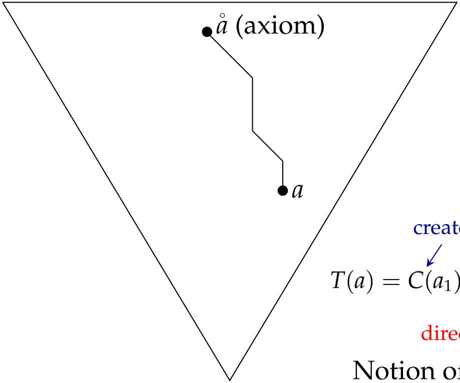

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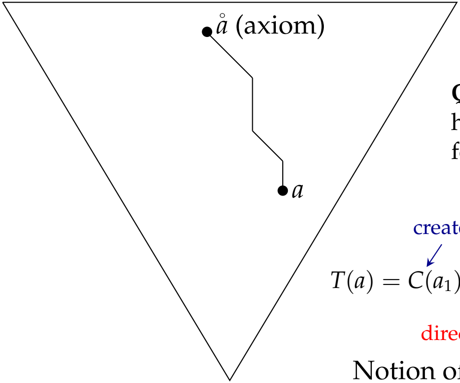

Question 1:
how do we get $\mathrm{L}(a) = \mathrm{R}(a)$ for all `app`-node a?

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Notion of **referent biposition**

THE PROBLEM OF COLLAPSE

$$T(a) = C(a_1)(y_1) \to \ldots \to C(a_p)(y_p) \to \mathrm{Hd}^p(T(\mathring{a}))$$

- Every $\mathtt{S}$-derivation P can be seen as a set of symbols, pointed by **bipositions** (a position points to a jugdment inside P, a biposition points to a type symbol ($\alpha, \to$) inside a jugdment inside P).
- Evey biposition (or so...) comes from a biposition in a type given in an axiom rule.
  Notion of **referent biposition** (set $\mathrm{ref}(P)$).
- In order to represent a $\mathscr{M}$-derivation Π by a $\mathtt{S}$-derivation P, we must associate to all axioms rules a parser $T(a)$ s.t. the **syntactic** equality $\mathrm{L}(a) = \mathrm{R}(a)$ holds for **every** application node.
- For now, let us just loosen the **synctactic equality** condition in the $\mathtt{app}$-rule.

HYBRID DERIVATIONS

Type system $\mathtt{S_h}$ is obtained from $\mathtt{S}$ by replacing the $\mathtt{app}$-rule by:

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- Each hybrid derivation P collapses into a $\mathscr{M}$-derivation Π.
- For any $\mathscr{M}$-derivation Π, easy to find a hybrid P s.t. $\overline{P} = \Pi$.

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- An **operable derivation** is a hybrid derivation endowed with a complete interface (for each `app`-rule).

Representation Lemma

Lemma

Let Π a $\mathscr{M}$-derivation typing t and a reduction sequence $\mathscr{R}$ (of length $\leqslant \omega$) and P a hybrid representative of Π.
Any reduction choice sequence along $\mathscr{R}$ can be built-in inside a complete interface for P.

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Intuition of the Proof:

- Consider a reduction sequence $t_0 \xrightarrow{b_0} t_1 \xrightarrow{b_1} t_2 \xrightarrow{b_2} \ldots$.

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- Consider a reduction sequence $t_0 \xrightarrow{b_0} t_1 \xrightarrow{b_1} t_2 \xrightarrow{b_2} \ldots$.
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REPRESENTATION LEMMA

Lemma

Let Π a $\mathscr{M}$-derivation typing t and a reduction sequence $\mathscr{R}$ (of length $\leqslant \omega$) and P a hybrid representative of Π.
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- Since each interface isomorphism of the reduced derivation is a residual an interface isomorphism, interface I_i can be lifted to P.

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Theorem

For all $\mathscr{M}$-derivation Π, there is a trivial $\mathtt{S}$-derivation P that collapses into Π.

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- At last, we notice that $\widetilde{\phi_a}(r_1) = r_2$ implies $\mathtt{ad}(r_1) < \mathtt{ad}(r_2)$ when r_1 occurs with a positive polarity.

Plan

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- Modify $\mathtt{S}$ to obtain a type theoretic characterization of **strongly normalizing (SN)** terms in Λ^{001} ?

Questions

Thank you for your attention !