Sequence Types for Hereditary Permutators

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TLCA Problem # 20

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with an intersection type system

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Today: using a coinductive type system to characterize hereditary permutators

t is a h.p. iff $\vdash t : \texttt{ptyp}$

Definition (Hereditary Permutators)

t is a **hereditary permutator (h.p.)**

$\Leftrightarrow$ t invertible in Scott's model

$\Leftrightarrow$ t invertible for $\beta\eta$-conversion w.r.t. composition ($\exists u,\ t \circ u =_{\beta\eta} u \circ t =_{\beta\eta} \mathtt{I}$)

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Characterization with Böhm trees

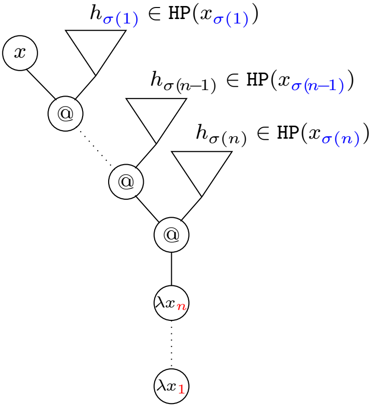
- For all $x \in \mathscr{V}$, the sets $\mathtt{HP}(x)$ of **x-headed Hereditary Permutators (x-HP)** ($x \in \mathscr{V}$) are defined by mutual coinduction:

$$\frac{h_1 \in \mathtt{HP}(x_1)\ \ldots\ h_n \in \mathtt{HP}(x_n) \quad (n \geqslant 0, \sigma \in \mathfrak{S}_n,\ x_i \neq x,\ x_i \text{ pairwise distinct}) \text{ and } h \to_{\mathtt{h}}^* \lambda x_1 \ldots x_n.x\, h_{\sigma(1)} \ldots h_{\sigma(n)}}{h \in \mathtt{HP}(x)}$$

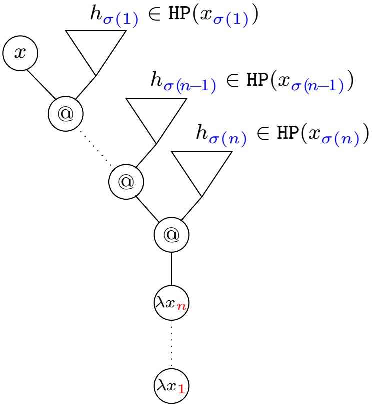
- t is a *(closed) hereditary permutator* iff $t \to_{\mathtt{h}}^* \lambda x.h$ with $h \in \mathtt{HP}(x)$ for some x.

See Barendregt, Chapter 21

An old problem. . .



head normal form of a x-h.p.



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Plan

1. Intersection types and sequences

2. Characterizing hereditary permutators

Intersection types (overview)

- Introduced by **Coppo-Dezani** (78-80) to “interpret more terms”
  - Charac. of Weak Norm. for λI-terms (no erasing β-step).
  - Extended later for λ-terms, head, weak or strong normalizatiion...
  - Filter models
- Model-checking
  - *Ong 06:* monadic second order (MSO) logic is decidable for higher-order recursion schemes (HORS)
  - *Kobayashi-Ong 09:* MSO is decidable for higher-order programs
    + using intersection types to simplify Ong's algorithm.
  - Refined by *Grellois-Melliès 14-15*
- Complexity analysis:
  - Upper bounds for reduction sequences (*Gardner 94, de Carvalho 07*) or exact bounds (*Bernadet-Lengrand 11, Accattoli-Lengrand-Kesner, ICFP'18*).
  - *Terui 06:* upper bounds for terms in a red. sequence
  - *De Benedetti-Ronchi della Roccha 16*: characterization of FPTIME

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Goal: Characterizing the set of h.p. with a *unique* type

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$\rightsquigarrow$ *e.g.*, the set of **infinitary WN** terms

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Step 1: characterize the set of h.p. in system $\mathtt{S}$

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t typable with $P \in \mathscr{P}$ iff t is a h.p.

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Step 2: give the set of h.p. a *unique* type

$\rightsquigarrow$ quotient $\mathscr{P}$ and verify everything is right

Why intersections types? (Coppo, Dezani, 1980)

Characterization in an intersection type system

Usually, equivalences of the form

"the program t is typable iff t is normalizing"

Idea: **several** certificates to a same subprogram (next slide).

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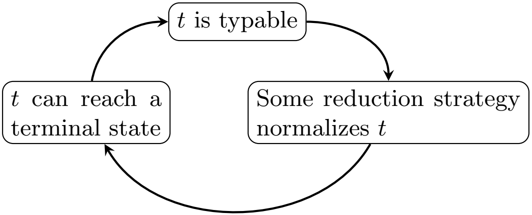

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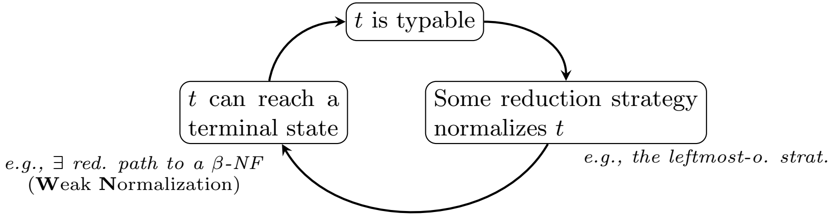

- Naively, $A \wedge B$ stands for $A \cap B$:

$$t \textit{ is of type } A \wedge B \textit{ if } t \textit{ can be typed with } A \textit{ as well as } B.$$

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Intuitions (syntax)

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- Intersection = kind of *finite polymorphism.*

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$(A \to A) \wedge ((A \to B) \to (A \to B))$ = **double** instance of $\forall X.X \to X$

(with $X = A$ and $X = A \to B$)

- But *less constrained*:

assigning $x : o \wedge (o \to o') \wedge (o \to o) \to o$ is legal.

*(**not** an instance of a polymorphic type except $\forall X.X :=$ `False`!)*

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Subject Reduction (SR):
Typing is stable under reduction.

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Typing is stable under anti-reduction.

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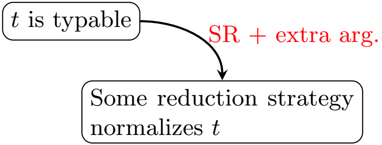

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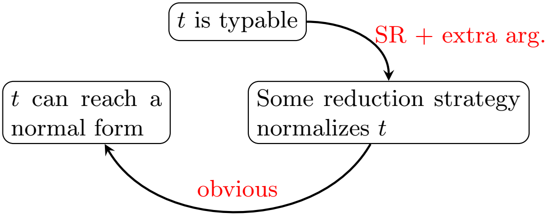

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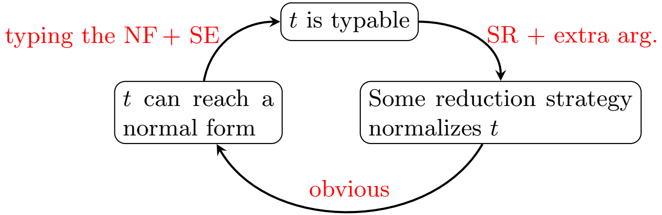

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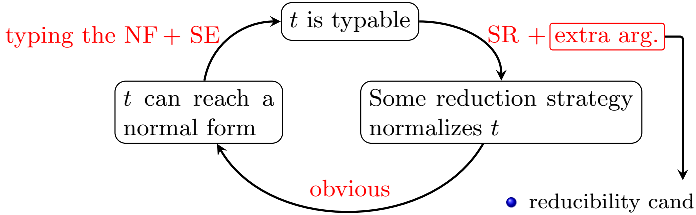

- reducibility cand.
- non-trivial well-founded order.
- non-idem. inter types: trivial
  size of derivation decreases during red.

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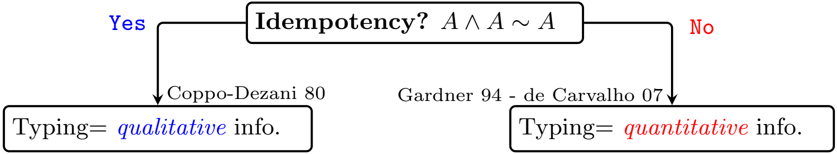

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Yes → Coppo-Dezani 80 → Typing= *qualitative* info.

No → Gardner 94 - de Carvalho 07 → Typing= *quantitative* info.

- Collapsing $A \wedge B \wedge C$ into $[A, B, C]$ (**multiset**) $\rightsquigarrow$ no need for perm rules etc.

$$A \wedge B \wedge A := [A, B, A] = [A, A, B] \neq [A, B] \qquad [A, B, A] = [A, B] + [A]$$

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- **Relevant** system (no weakening, *cf.* `ax`-rule)

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- **Non-idempotency ($\sigma \wedge \sigma \neq \sigma$):**
  in app-rule, pointwise multiset sum *e.g.*,

$$(x : [\sigma]; y : [\tau]) + (x : [\sigma, \tau]) = x : [\sigma, \sigma, \tau]; y : [\tau]$$

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but an arg. may be typed 0 time

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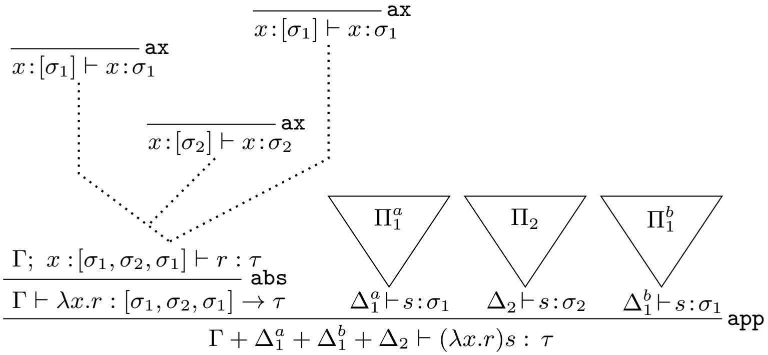

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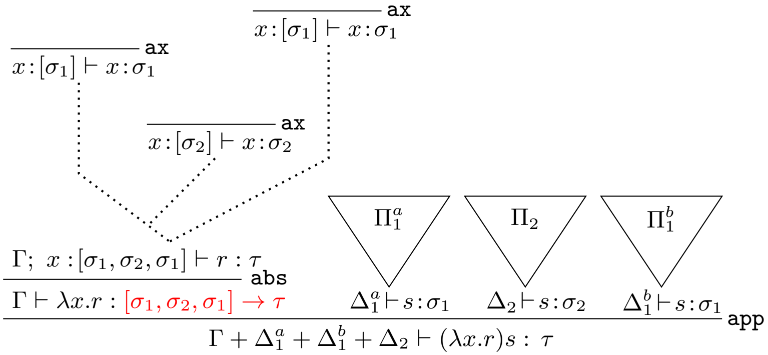

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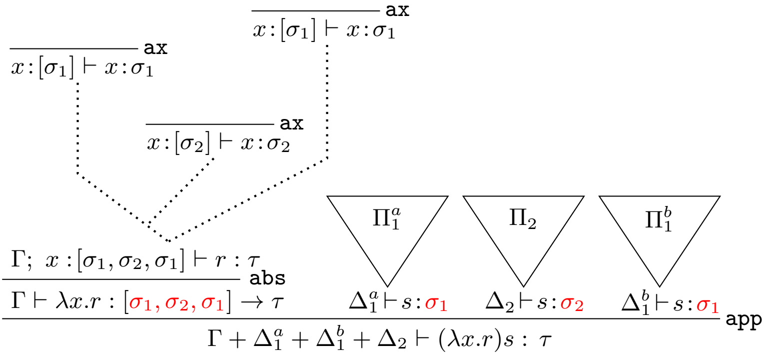

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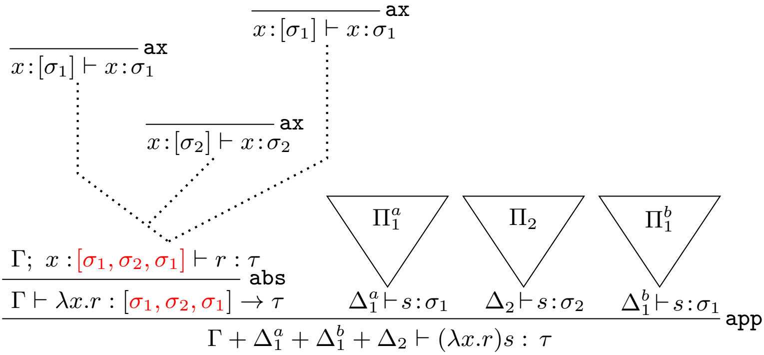

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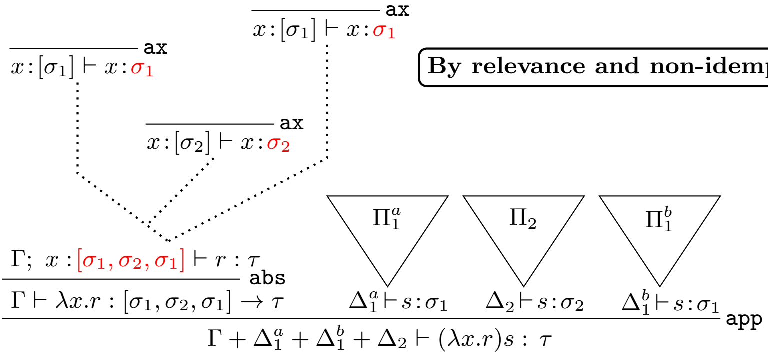
By relevance and non-idempotency!



Subject reduction and expansion in $\mathscr{R}_0$

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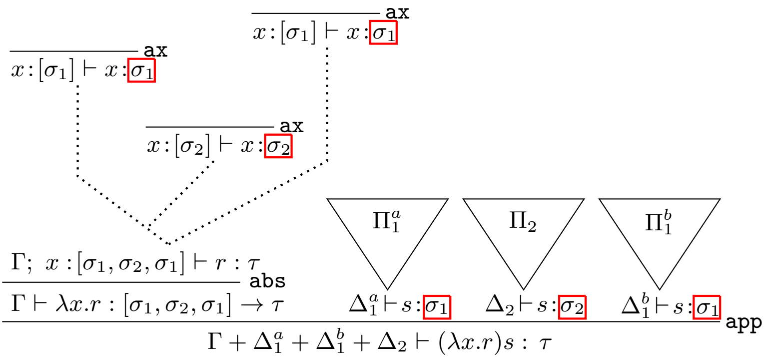

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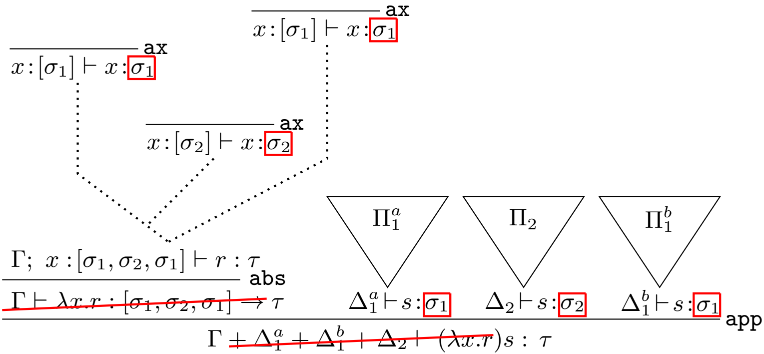

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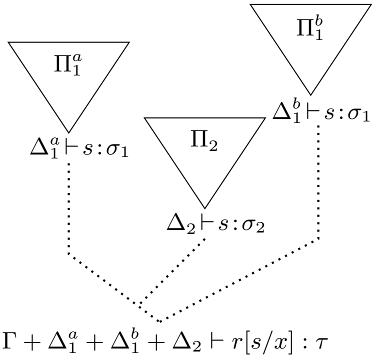

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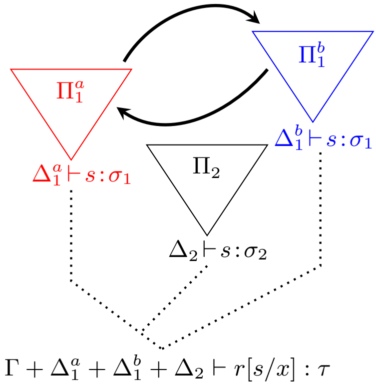

Non-determinism of SR

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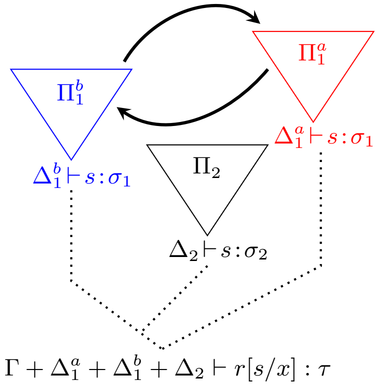

Non-determinism of SR

System $\mathtt{S}$ and sequence types [V., Lics'17]

System S

- coinductive type grammar
- replace $[\sigma_i]_{i\in I}$ with $(k\cdot\sigma_k)_{k\in K}$

sequence type (new intersection)

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$$(3 \cdot \sigma, 5 \cdot \tau, 9 \cdot \sigma) \ \textit{vs.} \ [\sigma, \tau, \sigma]$$

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Why system S?

- Coinduction necessary to *fully* type infinite NF
- Coinductive type grammar $\rightsquigarrow$ Ω is typable (unsoundness)
- Tracking necessary to recover soundness
  $\rightsquigarrow$ approximability (= validity criterion, next slide)

System S allows characterizing infinitary weak normalization

Approximability (intuitions)

- Order $\leqslant$ on the set derivations based on truncation

$$\dfrac{\overline{x : [[\ ] \to o] \vdash x : [\ ] \to o}}{x : [[o] \to o] \qquad \vdash x\, y : o}$$

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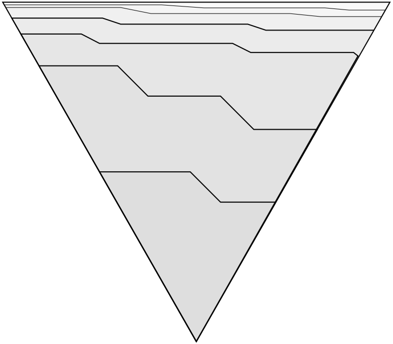

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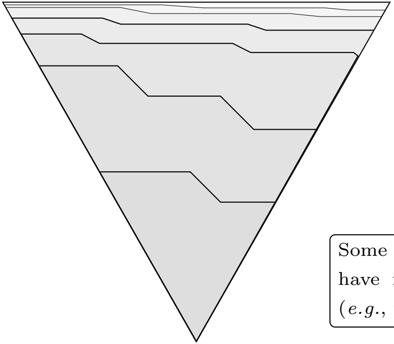


Some derivations do not have finite approximations (*e.g.*, typings of Ω)

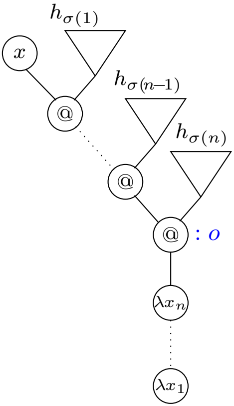
Properties of system S

Plan

Typing normal h.p. with permutator pairs

Goal:
Finding a set of pairs of types (S, T) s.t.
$x : S \vdash h : T$ when h is x-h.p.

Tracks are ignored above! (ok for NF)



h

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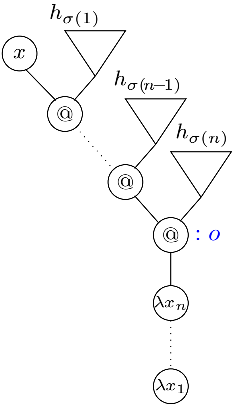

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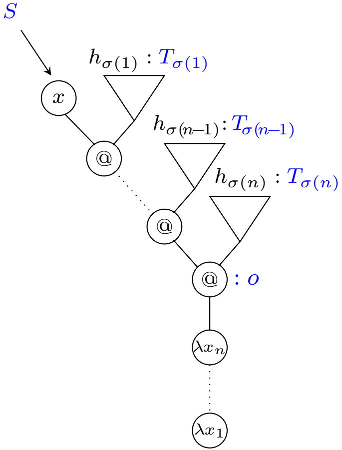

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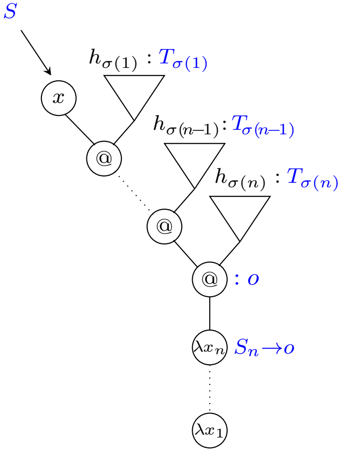

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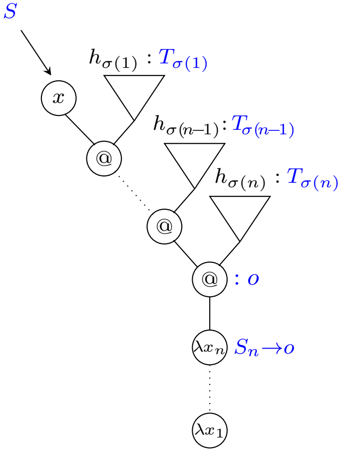

$h : T = S_1 \to \ldots \to S_n \to o$

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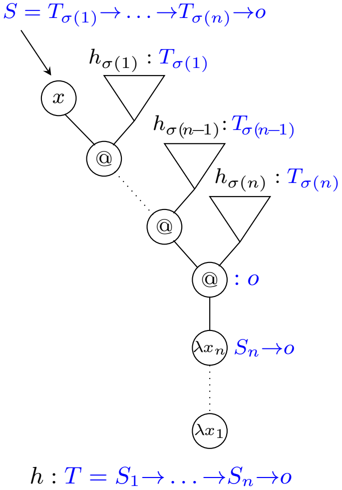

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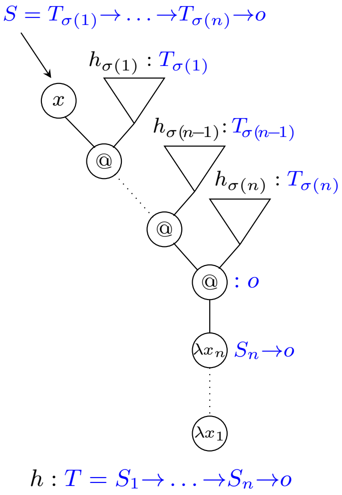

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Permutator pairs

- When o ranges over $\mathscr{O}$ (the set of type atoms), the set $\texttt{PP}(o)$ of **o-permutator pairs** (S, T), where S and T are $\texttt{S}$-types, is defined by mutual coinduction:

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Characterization of h.p. in system S

- t is a x-h.p. iff $x : (2 \cdot S) \vdash t : T$ for some $(S, T) \in \mathtt{PPP}$
- t is a h.p. iff $\vdash t : (2 \cdot S) \to T$ for some $(S, T) \in \mathtt{PPP}$.

- For ∞-NF: $\Rightarrow$ (animation) $\Leftarrow$: use properness
- non-NF: use ∞-subj. reduction and expansion

Characterization in system S

- t is a x-h.p. iff $x : (2{\cdot}S) \vdash t : T$ (with (S, T) perm. pair)
- t is a h.p. iff $\vdash t : (2{\cdot}S) \rightarrow T$

Proof.

- NF case: previous slide
- Use infinitary subject reduction and expansion

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Does this preserve type soundness/completeness?

Permutator schemes of degree d

Let $d \in \mathbb{N}$. A x-**permutator scheme** of degree d is a term t whose Böhm tree is equal to that of a x-h.p. for applicative depth $< d$.

Applicative depth: number of nestings inside arguments.

- Any term t is a 0-p.s.
- $h = \lambda x\, x_1 x_2.(x\,(\lambda x_{1,1} x_{1,2}.x_2\, t_1\, t_2))(\lambda x_{2,1}.x_1\, t_3)$ is a 2-p.s. ($t_{1,2,3}$:terms)

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Proper permutator pairs of degree d

$\mathtt{PPP}_d$: truncations $((S)^{\leqslant d}, (T)^{\leqslant d})$ of a proper permutator pair at domain depth $< d$ (finite types!)

Domain depth: number of nestings inside left-hand sides of arrows

Truncation

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t is a d-p.s. headed by x iff $x : (2{\cdot}S) \vdash t : T$ (with $(S, T) \in \mathtt{PPP}_d$)

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Compatible truncation

$x : (2 \cdot S) \vdash t : T$ iff, for all $d \in \mathbb{N}$, $x : (2 \cdot (S)^{\leqslant d}) \vdash t : (T)^{\leqslant d}$

System $\mathtt{S_{hp}}$ = System $\mathtt{S}$ +

"Infinitary rule":

$$\frac{x : (2 \cdot S) \vdash t : T \qquad (S,T) \in \mathtt{PPP}}{\vdash \lambda x.t : \mathtt{ptyp}}\,\mathtt{hp}$$

"Finitary" rule (level d for all d):

$$\frac{x : (2 \cdot S) \vdash t : T \qquad (S,T) \in \mathtt{PPP}_d}{\vdash \lambda x.t : \mathtt{ptyp}_d}\,\mathtt{hp}_d$$

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Approximability extended with the rule
$\mathtt{ptyp}_d \leqslant \mathtt{ptyp}$ for all d

System $\mathtt{S}_{\mathtt{hp}}$

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"Finitary" rule (level d for all d):

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Dynamic properties and final characterization

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From ∞-s.r. and s.e. + inversion for NF + truncation:

A unique type for hereditary permutators

For all terms t, t is a h.p. iff $\vdash t : \mathtt{ptyp}$ in system $\mathtt{S}_{\mathrm{hp}}$.

Summary and future work

Contribution: characterizing hereditary permutators with a unique type

- Not possible in the inductive case (not r.e.)
- System $\mathtt{S}$: coinductive variant of **non-idempotent intersection** types:
  - elements of multisets annotated with **tracks**
  - allows recovering soundness w. **approximability** (=*validity criterion*).
- In system $\mathtt{S}$, h.p. characterized with **proper permutator pairs**.

- System $\mathtt{S}_{\mathtt{hp}}$: **"collapsing"** proper permutator pairs using constant $\mathtt{ptyp}$.
  $\mathtt{ptyp}$ approximated by $\mathtt{ptyp}_d \rightsquigarrow$ approximability extends to $\mathtt{S}_{\mathtt{hp}}$.
- Truncation lemma for t NF:

$$\vdash t : \mathtt{ptyp} \text{ approximated by } \vdash t : \mathtt{ptyp}_d \text{ (for all } d \in \mathbb{N})$$

- $\mathtt{S}_{\mathtt{hp}}$: infinitary subject reduction and expansion
  $\rightsquigarrow$ retrieving methods of finitary intersection type systems
- Soundness and completeness w.r.t. h.p.: t is a h.p. iff $\vdash t : \mathtt{ptyp}$ in $\mathtt{S}_{\mathtt{hp}}$

Future work: Other sets of terms with infinitary behaviors

inc. normalization in other ∞-calculi