Coinductive Intersection Types are Completely Unsound

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- Another use of denotations: **equating** or **separating programs** *i.e.* two states that have different denotations cannot be instances of the same program.

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- Typing: assigning formulas (called *types*) to variables.
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- When a type system enjoys **subject reduction** and **expansion**, types are execution invariants (and they usually provide us with models of λ-calculus).

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- Some programs are non terminating but **productive**.
- Many possible definitions or variants of sound non termination
  Klop and alii[95], Endrullis,Polonsky and alii[15]

CONTRIBUTIONS

Using type theory, we build a **completely unsound** type system and a **non-sensible model** of pure λ-calculus in which:

- Every term has a non-empty denotation (including the **mute terms**).
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Related works

- Jacopino[75]: **easy** terms (t is easy if it can be consistently equated to any other term)
- Berarducci[96]: **mute** terms (”The most undefined terms”).
- Bucciarelli,Carraro,Favro,Salibra[15]: *Graph easy Sets of mute lambda terms*, TCS.

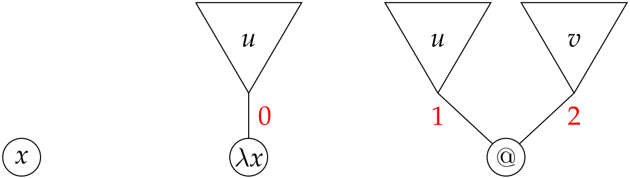
Plan

Types and Relevance

Coinductive Types

System S (sequential intersection)

λ-TERMS AS LABELLED TREES



Variable x **Abstraction** $\lambda x.u$ **Application** $u\,v$

λ-terms as labelled trees

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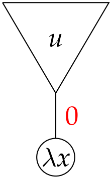

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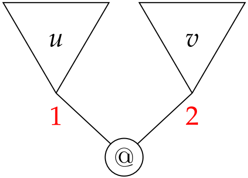
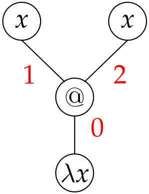


Application $u\,v$

$$\Delta = \lambda x.xx$$



$\lambda x.xx$

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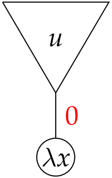

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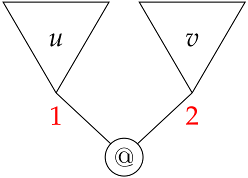

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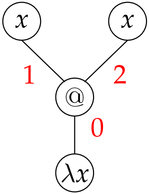
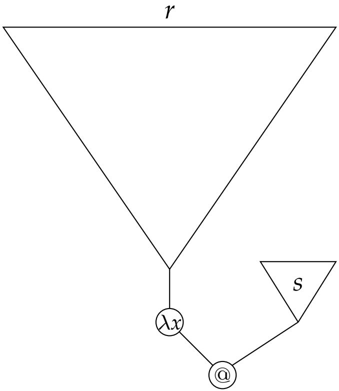


$\lambda x.xx$

Position: finite sequence in $\{0,1,2\}^*$, *e.g.* $0 \cdot 0 \cdot 2 \cdot 1 \cdot 2$.

β-REDUCTION



Redex:

$(\lambda x.r)s$

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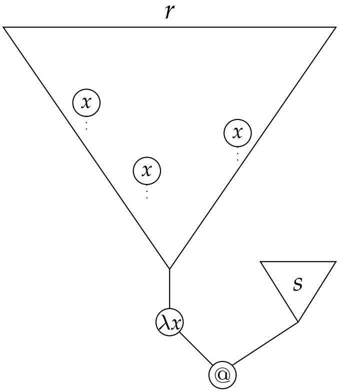

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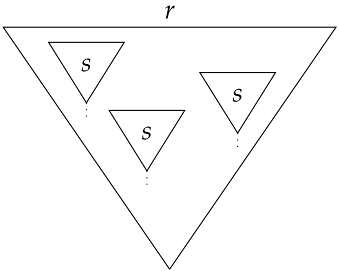
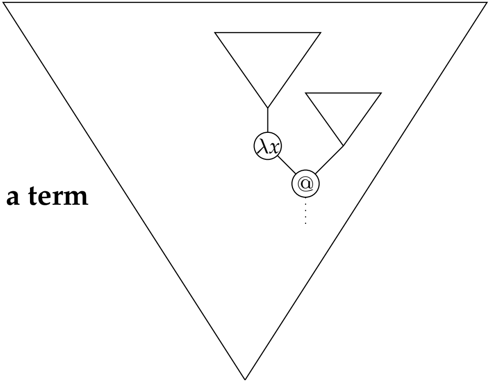


Reduct:

$r[s/x]$

β-REDUCTION

Redex
nested in a term



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- If a Type System is able to type a non (weakly) head-normalizing term (*e.g.* $\Omega = \Delta\,\Delta$), it is said here to be **unsound**.

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- If a Type System is able to type every term, it is said to be **completely unsound**.
- With SR and SE, a completely unsound type system should yield a model for pure λ-calculus.

TYPING RULES OF $\mathscr{R}_0$ (GARDNER/DE CARVALHO)

Types (τ, σ_i): $\tau, \sigma_i := \ o \in \mathscr{O} \mid [\sigma_i]_{i \in I} \to \tau.$

Context (Γ, Δ): assign *intersection* types to variables.

$$\frac{}{x : [\tau] \vdash x : \tau}\ \mathtt{ax} \qquad \frac{\Gamma, x : [\sigma_i]_{i\in I} \vdash t : \tau}{\Gamma \vdash \lambda x.t : [\sigma_i]_{i\in I} \to \tau}\ \mathtt{abs}$$

$$\frac{\Gamma \vdash t : [\sigma_i]_{i\in I} \to \tau \qquad (\Delta_i \vdash u : \sigma_i)_{i\in I}}{\Gamma +_{i\in I} \Delta_i \vdash t\,u : \tau}\ \mathtt{app}$$

Examples:

$$\cfrac{\cfrac{}{x : [\tau] \vdash x : \tau}\ \mathtt{ax}}{\vdash \lambda x.x : [\tau] \to \tau}\ \mathtt{abs} \qquad \cfrac{\cfrac{}{x : [\tau] \vdash x : \tau}\ \mathtt{ax}}{x : [\tau] \vdash \lambda y.x : [\,] \to \tau}\ \mathtt{abs}$$

Relevance vs Irrelevance

- **Observation:** In system $\mathscr{R}_0$, $\lambda x.x$ (resp. $\lambda y.x$) can only be typed with a type of the form $[\tau] \to \tau$ (resp. $[\,] \to \tau$).

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- In $\mathscr{R}_{0,\texttt{w}}$, we may derive:

$$\frac{\dfrac{}{x : [\tau, \tau_1, \tau_1] \vdash x : \tau}\ \texttt{axw}}{\vdash \lambda x.x : [\tau, \tau_1, \tau_2] \to \tau}\ \texttt{abs} \qquad \frac{\dfrac{}{x : [\tau], y : [\tau] \vdash x : \tau}\ \texttt{axw}}{x : [\tau] \vdash \lambda y.x : [\tau] \to \tau}\ \texttt{abs}$$

Subject Reduction Property for $\mathscr{R}_0$

If $\Pi \rhd \Gamma \vdash t : \tau$ and $t \to t'$, then $\exists \Pi' \rhd \Gamma \vdash t' : \tau$

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$$(\lambda x.r)s \to r[s/x]$$

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Axiom leaves typing x inside Π_r

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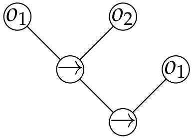
System S (sequential intersection)

INDUCTIVE VS COINDUCTIVE TYPES

Examples with Simple Types

Inductive type:

$o_1 \to o_2 \to o_1$



Coinductive type:

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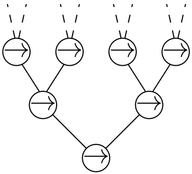

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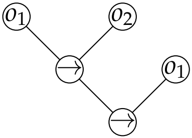

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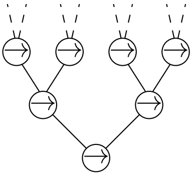

$A_{\mathtt{ref}}$ is a **reflexive type**.

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  Proof.

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Question: what is the set of typable terms in $\mathscr{R}$?

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We study then **typability** as a first order theory. For that, we resort to another type system $\mathtt{S}$, in which features *pointers*.
System $\mathtt{S}$ collapses on $\mathscr{R}$. Thus, if every term is typable in $\mathtt{S}$, then every term is typable in $\mathscr{R}$.

Plan

Types and Relevance

Coinductive Types

System S (sequential intersection)

SEQUENTIAL INTERSECTION

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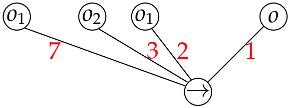
- *Example:* $(7 \cdot o_1, 3 \cdot o_2, 2 \cdot o_1) \to o$



DERIVATIONS OF S

The set `Deriv` of rigid derivations is *coinductively* generated by:

$$\frac{}{x : (k \cdot T) \vdash x : T}\ \texttt{ax} \qquad \frac{C;\, x : (S_k)_{k \in K} \vdash t : T}{(S_k)_{k\in K} \vdash \lambda x.t : C(x) \to T}\ \texttt{abs}$$

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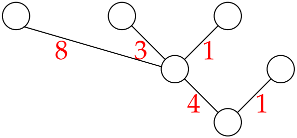
- S features **pointers** called **bipositions**.

CANDIDATE SUPPORTS

What is a correct type ?



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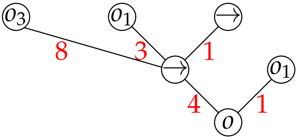

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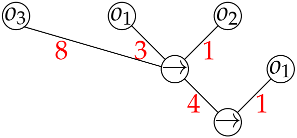

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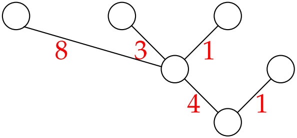

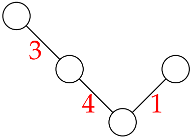

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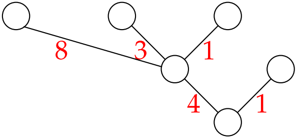

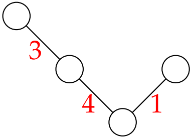

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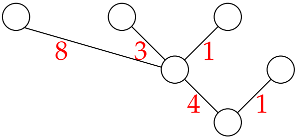

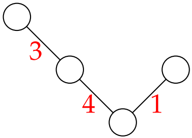

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Candidate Support: a set of positions that is the support of a type

- $c \rightarrow_{\mathtt{t1}} c \cdot k$ (a candidate supp is a tree)
- $c \cdot 1 \rightarrow_{\mathtt{t2}} c \cdot k$ (if a node does not have a 1-son, it is a leaf)

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- Then, we show that there is a *non-empty* set that satisfies them.

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- $(a,\, c) \to_{\mathtt{asc}} (a \cdot 1, 1 \cdot c)$ if $t(a) = @$.
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- $(a, c) \to_{\mathtt{t1}} (a, c \cdot k)$.
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This works for the infinitary λ-calculus.

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Theorem: This yields a non-sensible model that discriminates terms according to their order.

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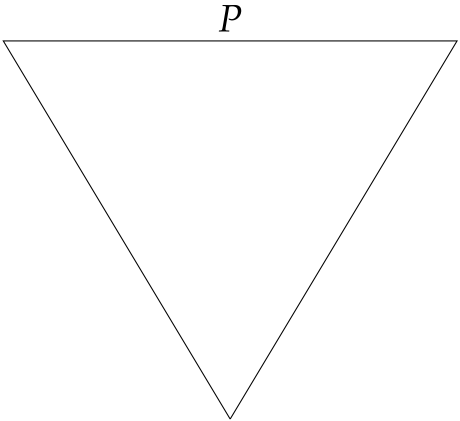
- Equational theory of the Model.
- Is the collapse of $\mathscr{R}$ onto $\mathscr{D}$ (idempotent intersection) surjective ?

QUESTIONS

Thank you for your attention !

Pointers



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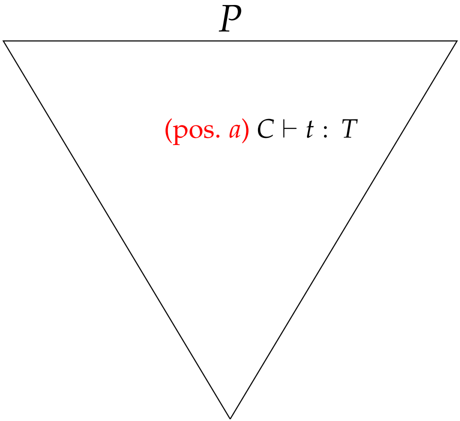

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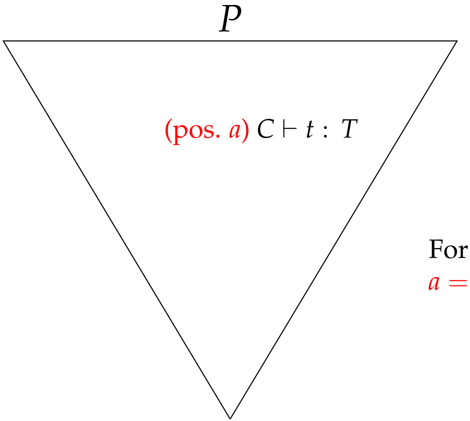
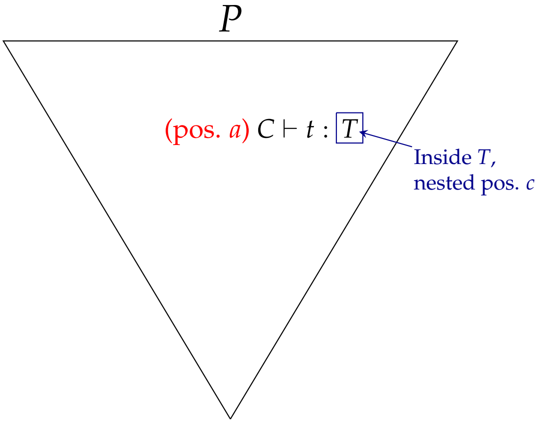


For instance
$a = 0 \cdot 1 \cdot 3 \cdot 0 \cdot 8 \cdot 1$

POINTERS



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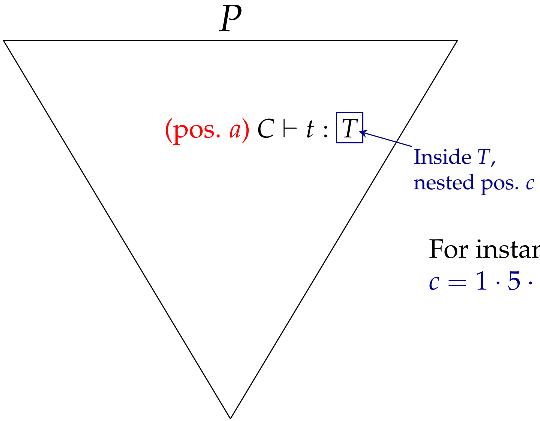

For instance
$c = 1 \cdot 5 \cdot 3 \cdot 1 \cdot 4$

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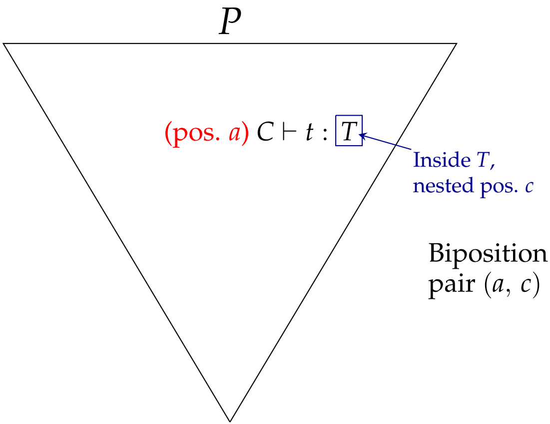

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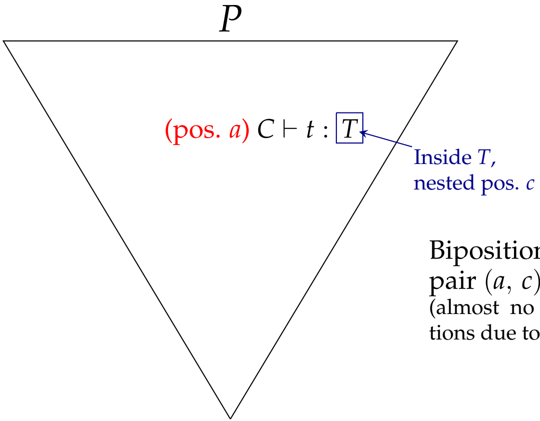

Biposition (right h.s.):
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(almost no need for left bipositions due to relevance)

Bisupport of P: the set of (right or left) bipositions

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Some bipositions can be intuitively identified in a derivation.

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Two occurrences of the same type

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$$\frac{}{x : (k \cdot T) \vdash x : T}\ \texttt{ax} \qquad\qquad \frac{C \vdash t : T}{C; (S_k)_{k \in K} \vdash \lambda x.t : C(x) \to T}\ \texttt{abs}$$

Let $k \geqslant 2$. We have two cases :

$$\frac{C; x : (S_k)_{k \in K} \vdash t : T \qquad \text{(pos. } a \cdot 0\text{)}}{C \vdash \lambda x.t : (S_k)_{k \in K} \to T \qquad \text{(pos. } a\text{)}}$$

Look at S_7
inside this seq. type.

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We then set: $(a,7\cdot c)\to_{\mathtt{pi}}(a',c)$ when $t(a)=\lambda x$

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We then set: $(a, 7 \cdot c) \to_{\texttt{pi}} \texttt{b}_\perp$ when $t(a) = \lambda x$

REFERENTS

- Let $\equiv$ be the reflexive, transitive, symmetric closure of $\rightarrow_{\mathtt{asc}} \cup \rightarrow_{\mathtt{pi}}$.

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Referents

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  Then a derivation P typing t holds $\mathtt{b}_1$ iff it holds $\mathtt{b}_2$.
- Moreover, P cannot hold $\mathtt{b}_\perp$.
- An equivalence class of $\equiv$ is called a **referent**.
  Let $\mathtt{Ref}$ be the quotient set defined by $\equiv$.
  We write $\mathtt{b} : \mathtt{r}$ or $\mathtt{r} : \mathtt{b}$.

Consumption

CONSUMPTION

$$\frac{C \vdash t : (S_k)_{k \in K} \to T \quad (\text{pos. } a \cdot 1) \qquad (D_k \vdash u : S_k \ (\text{pos. } a \cdot k)\ )_{k \in K}}{C \cup_{k \in K} D_k \vdash tu : T \quad (\text{pos. } a)}$$

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We then set:
$(a \cdot 1,\ k \cdot c) \xrightarrow{a} (a \cdot k,\ c)$ when $t(a) = @$