Infinitary Intersection Types as Sequences

(A New Answer to Klop's Problem)

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IRIF, Paris 7

Elica meeting, Bologna

October 7, 2016

PLAN

HEREDITARY HEAD-NORMALIZATION

- **Head Normal Forms (HNF):** terms t of the form:

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- A term is **head-normalizing (HN)** if it can be reduced to a HNF (in a finite number of steps)

- Coinductively, a term is **hereditary head-normalizing (HHN)** if it can be reduced to a HNF and all the head arguments are themselves HHN.

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- Can a coinductive ITS characterize the set of HHN terms?

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PURPOSES OF THIS TALK

- Present the key notions of **truncations** and **approximability** (meant to avoid *irrelevant* derivations).
- Understand why **commutative intersection** (here, Gardner/de Carvalho's **multiset intersection**) is **unfit** to express those key notions.
- Present the coinductive type assignment system $\mathtt{S}$: intersection types are **sequences** of types, instead of *sets* of types (idempotent intersection fw.) or *multisets* of types (regular non-idempotent fw.).

Plan

Typing Rules of $\mathscr{M}_0$ (Gardner/de Carvalho)

Types (τ, σ_i): $\tau, \sigma_i := \alpha \in \mathscr{X} \mid [\sigma_i]_{i\in I} \to \tau$.

Context (Γ, Δ): assigns *intersection* types to variables.

$$\frac{}{x : [\tau] \vdash x : \tau}\ \text{ax} \qquad\qquad \frac{\Gamma, x : [\sigma_i]_{i\in I} \vdash t : \tau}{\Gamma \vdash \lambda x.t : [\sigma_i]_{i\in I} \to \tau}\ \text{abs}$$

$$\frac{\Gamma \vdash t : [\sigma_i]_{i\in I} \to \tau \qquad \left(\Delta_i \vdash u : \sigma_i\right)^{i\in I}}{\Gamma + \sum_{i\in I} \Delta_i \vdash t(u) : \tau}\ \text{app}$$

Remark

- Multiset equality: $[\sigma, \tau, \sigma] = [\sigma, \sigma, \tau] \neq [\sigma, \tau]$
- Multiplicative rules: accumulation of typing information .
- Possibility to **forget** the argument (empty multiset).

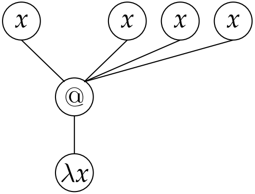
ALTERNATIVE PRESENTATION

Standard presentation

$$\dfrac{\dfrac{\dfrac{}{x : [[\alpha,\beta,\alpha]\to\alpha] \vdash x : [\alpha,\beta,\alpha]\to\alpha}\,\text{ax} \quad \dfrac{}{x:[\alpha]\vdash x:\alpha}\,\text{ax} \quad \dfrac{}{x:[\beta]\vdash x:\beta}\,\text{ax} \quad \dfrac{}{x:[\alpha]\vdash x:\alpha}}{x : [\alpha,\beta,\alpha,[\alpha,\beta,\alpha]\to\alpha] \vdash xx : \alpha}}{\vdash \lambda x.xx : [\alpha,\beta,\alpha,[\alpha,\beta,\alpha]\to\alpha]\to\alpha}\,\text{abs}$$

ALTERNATIVE PRESENTATION

Alternative presentation



$\lambda x.xx$

- Indicate the arity of application rules.

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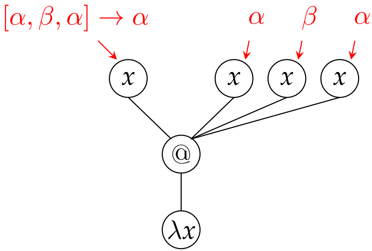

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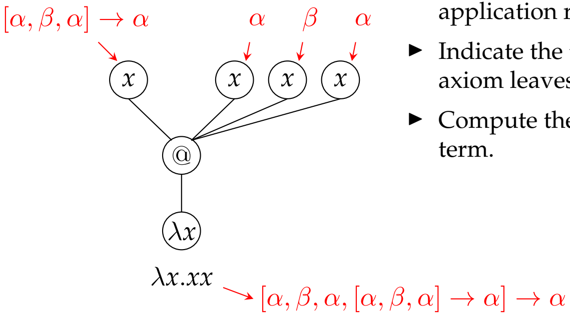

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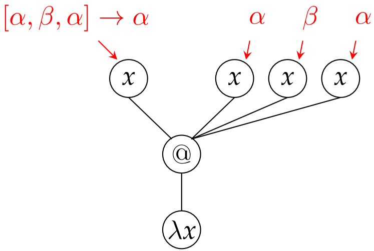

$\lambda x.xx$ $\to$ $[\alpha, \beta, \alpha, [\alpha, \beta, \alpha] \to \alpha] \to \alpha$

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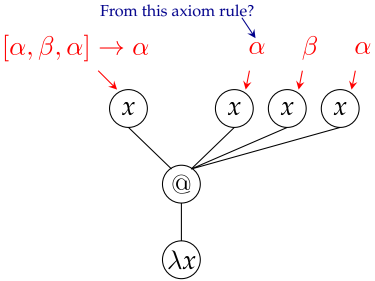

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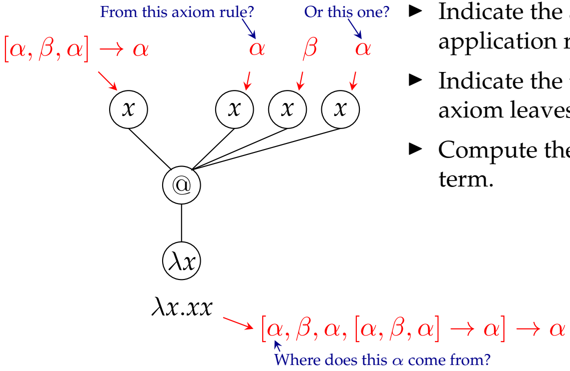

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If $\Pi \rhd \Gamma \vdash t : \tau$ and $t \to t'$, then $\exists \Pi' \rhd \Gamma \vdash t' : \tau$

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$$(\lambda x.r)s \to r[s/x]$$

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Axiom leaves
typing x inside Π_r

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Vocabulary:
We say each **association** (between x-axiom leaves and arg-derivations) yields a **derivation reduct** Π' typing $r[s/x]$.

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Observation:
If a type σ occurs several times in $[\sigma_i]_{i \in I}$, there can be several associations, each one yielding a possibly different derivation reducts Π'.

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A judgement $\Gamma \vdash t : \tau$ is **unforgetful** if there is no negative occurrence of $[\,]$ in Γ and no positive occurrence of $[\,]$ in τ.

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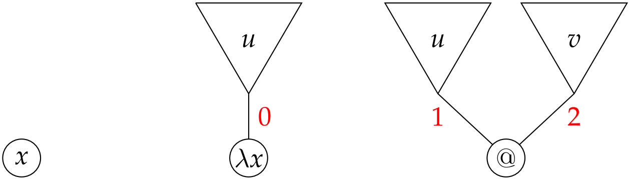
- $[\,]$ occurs negatively in $[\,] \to \tau$
- If $[\,]$ occurs negatively in σ_2 then $[\,]$ occurs positively in $[\sigma_1, \sigma_2, \sigma_3] \to \tau$ and so on.

PLAN

∞-TERMS



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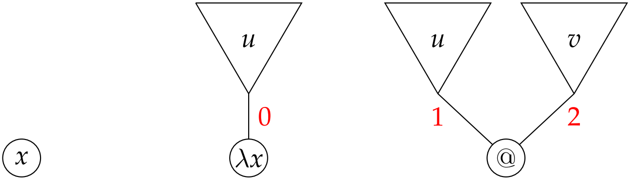

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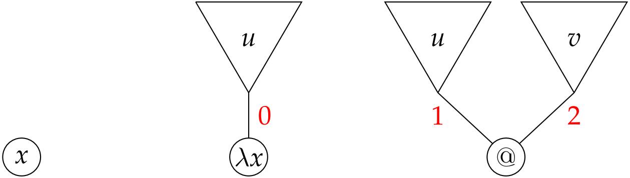

Variable x **Abstraction** $\lambda x.u$ **Application** $u\,v$

- **Position**: finite sequence in $\{0, 1, 2\}^*$, *e.g.* $0 \cdot 0 \cdot 2 \cdot 1 \cdot 2$.
- **Applicative Depth (a.d.):** number of $\nearrow$-edges *e.g.*

$$\mathtt{ad}(1 \cdot 2 \cdot 2 \cdot 0 \cdot 2 \cdot 1 \cdot 2) = 4$$

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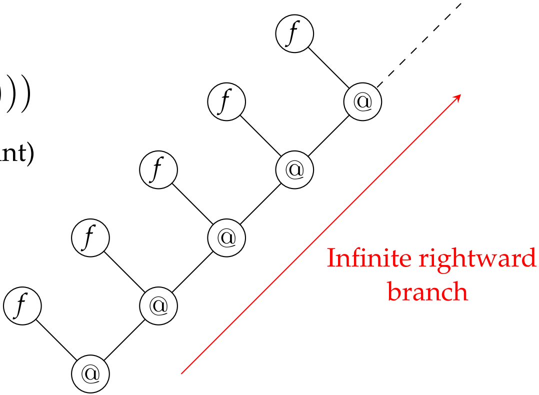
$$f^\omega := f(f(f(\ldots)))$$

i.e. $f^\omega = f(f^\omega)$ (fixpoint)



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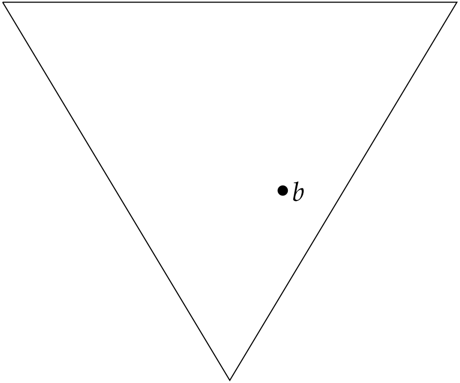

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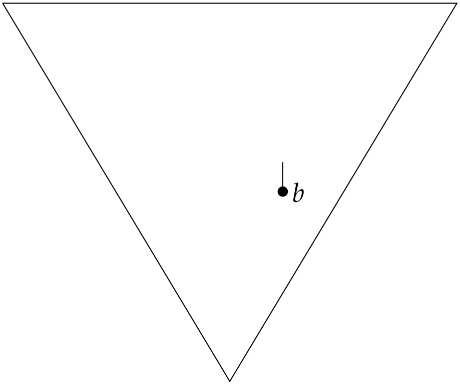

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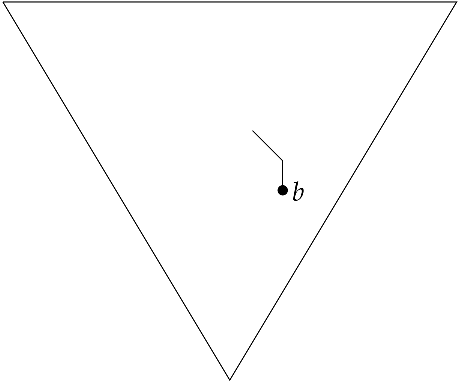

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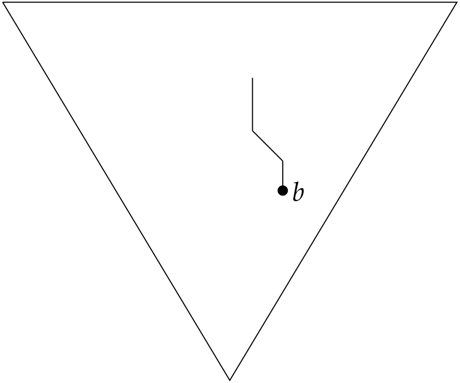

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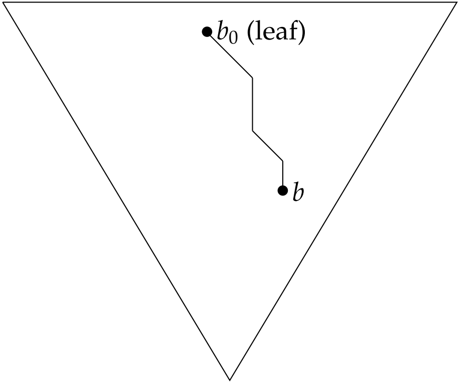

- Start from $b \in \mathtt{supp}(t)$
- Move $\uparrow$ or $\nwarrow$
- A leaf b_0 must be reached

STRONG CONVERGENCE

Definition

A reduction sequence $t_0 \xrightarrow{b_0} t_1 \xrightarrow{b_1} t_2 \xrightarrow{b_2} \ldots \xrightarrow{b_{n-1}} t_n \xrightarrow{b_n} \ldots$ is **strongly converging** if it is of finite length or if $\lim \mathtt{ad}(b_n) = \infty$.

STRONG CONVERGENCE

$$\Delta_f := \lambda x.f(xx) \qquad \Delta_f\Delta_f\text{: "Curry"}$$

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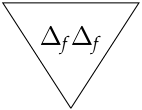

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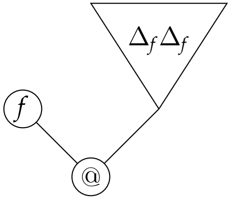

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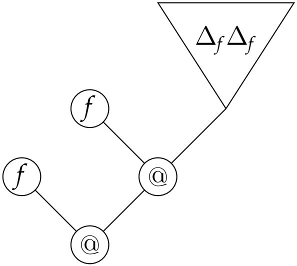

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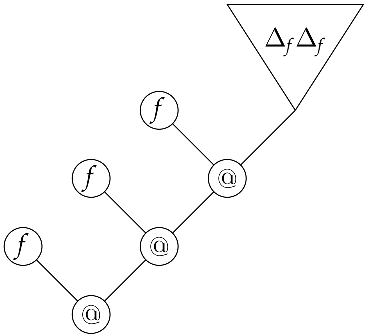

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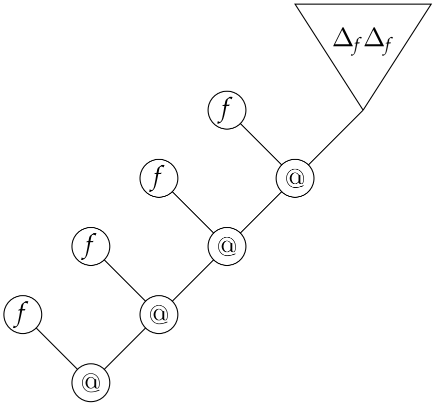
$$\Delta_f\Delta_f \to f(\Delta_f\Delta_f) \to f^2(\Delta_f\Delta_f) \to f^3(\Delta_f\Delta_f) \to f^4(\Delta_f\Delta_f) \to \ldots \to^\infty f^\omega$$



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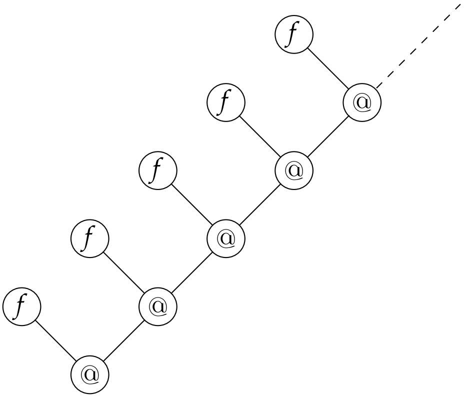

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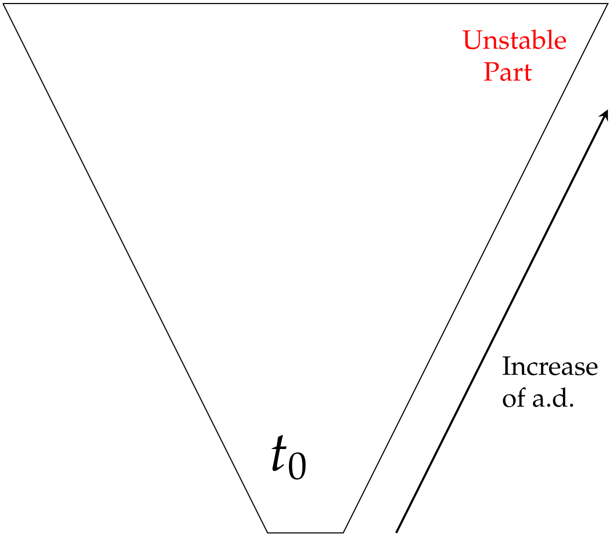

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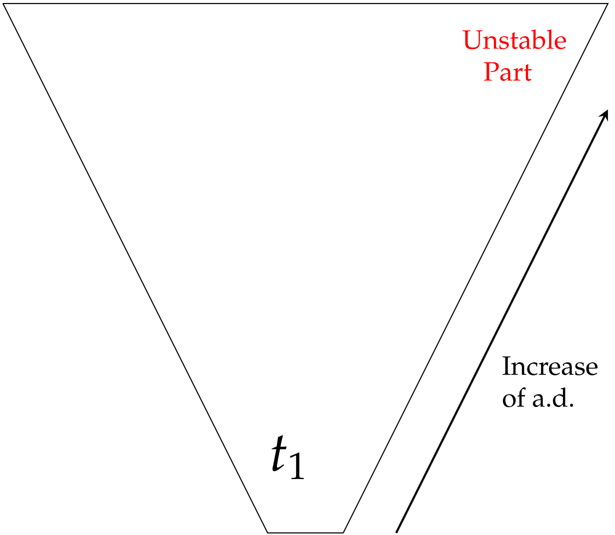

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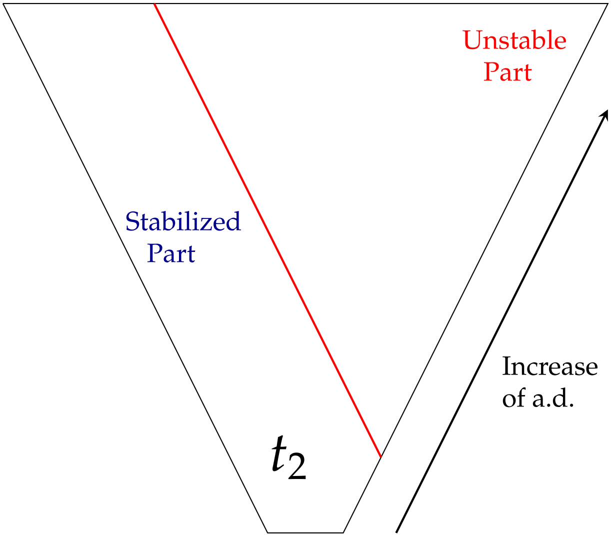

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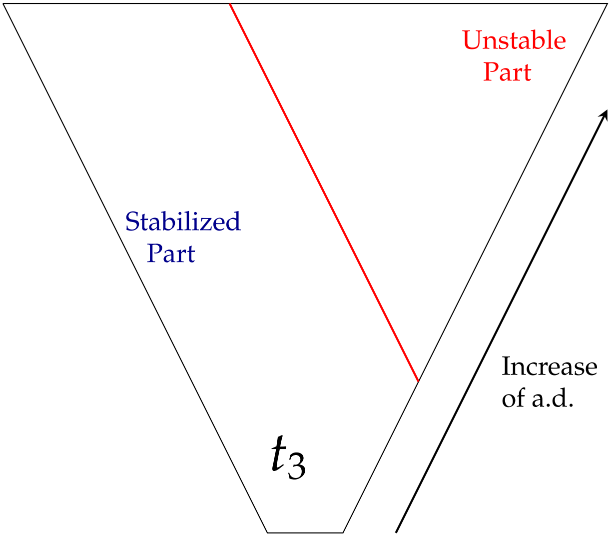

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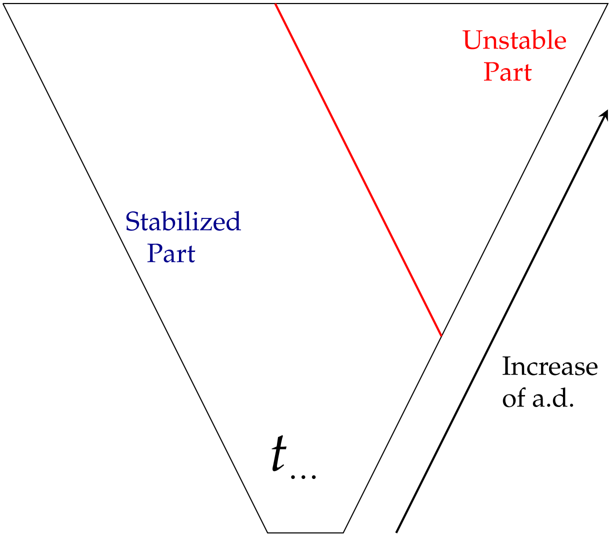

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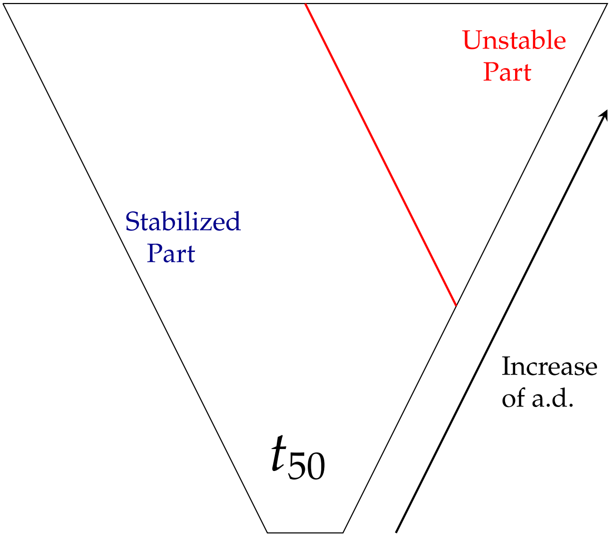

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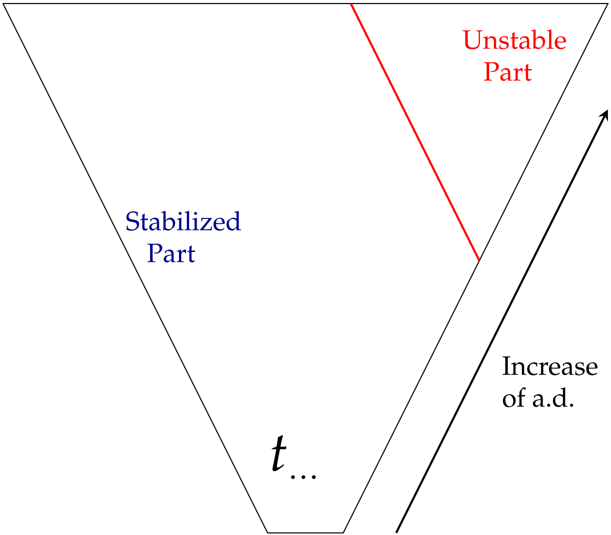

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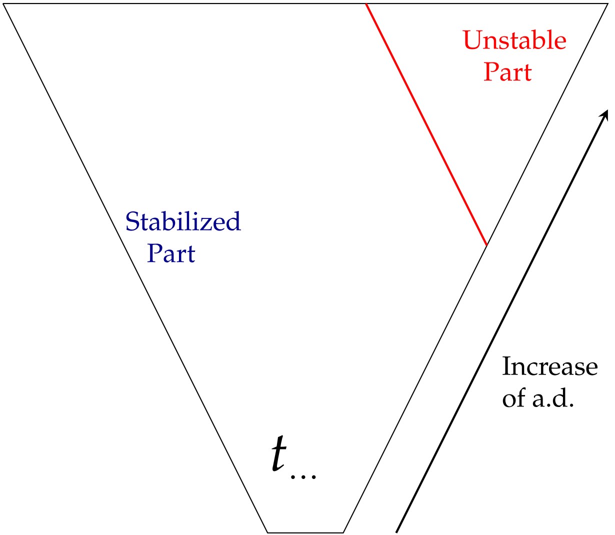

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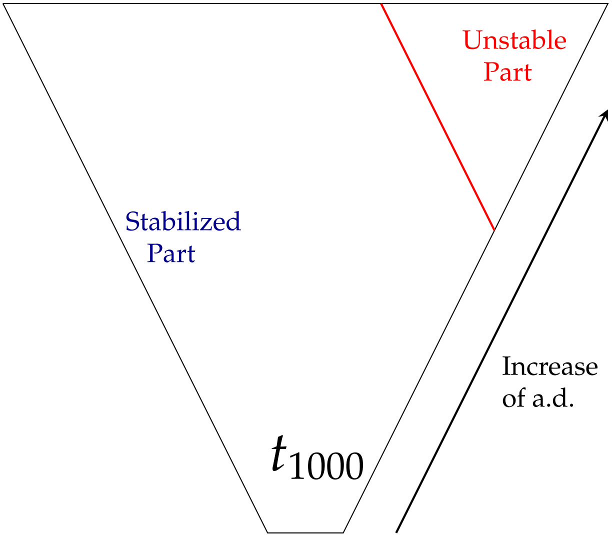

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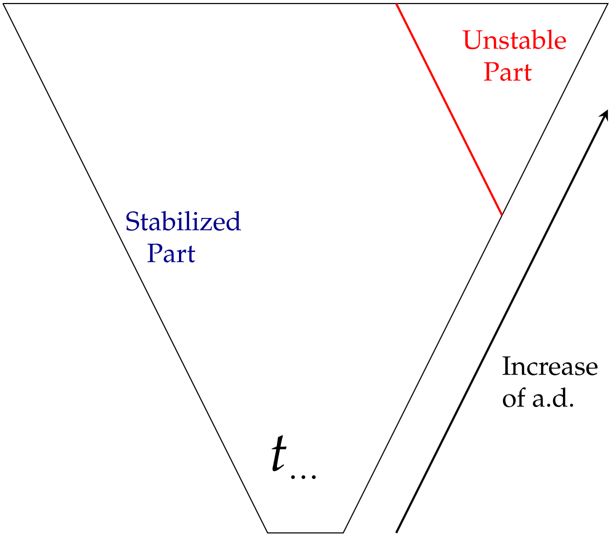

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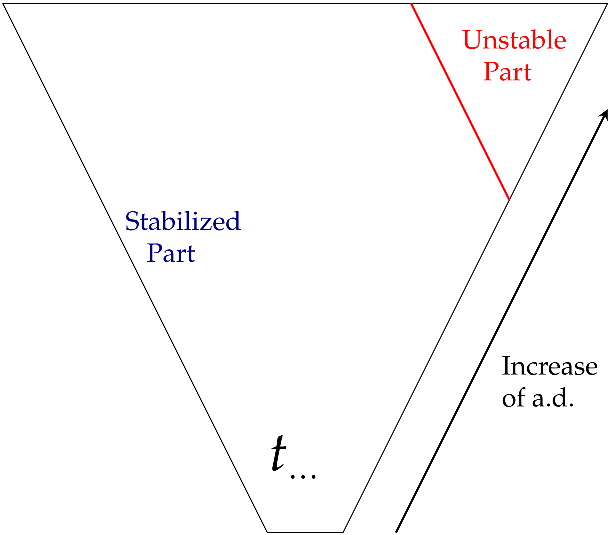

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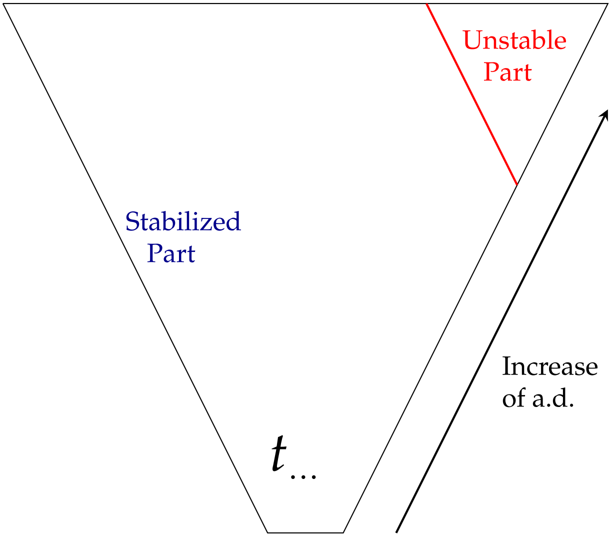

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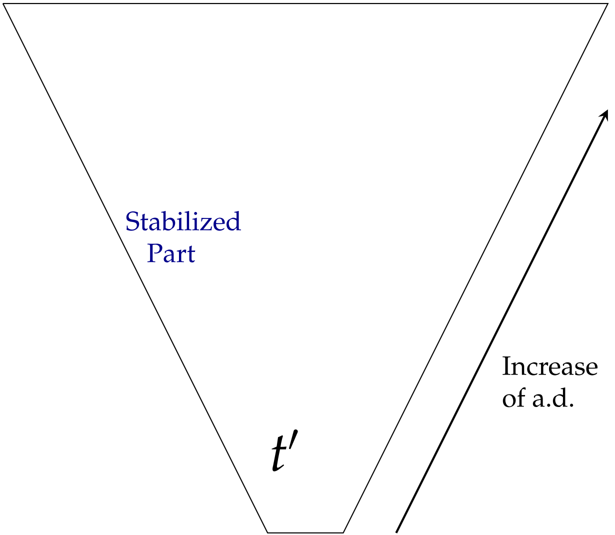

STRONG CONVERGENCE

Conclusion

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A **strongly converging reduction sequence (s.c.r.s)** allows us to define its **limit**.

Infinitary Normalization

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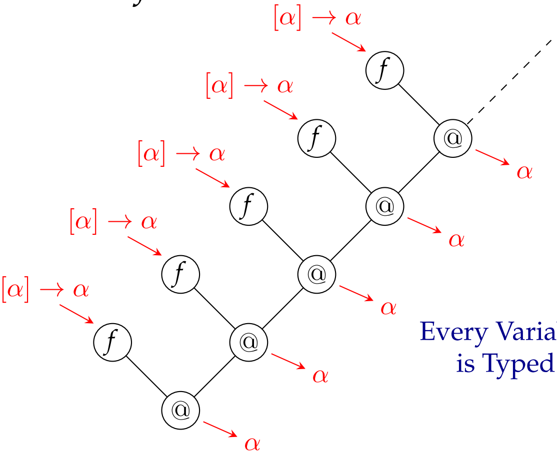
- Thus, a (finite) term is HHN iff it is 001-WN.

PLAN

TRUNCATION (FIGURES)

$$\Pi' \triangleright \Gamma \vdash f^\omega : \alpha$$



Every Variable
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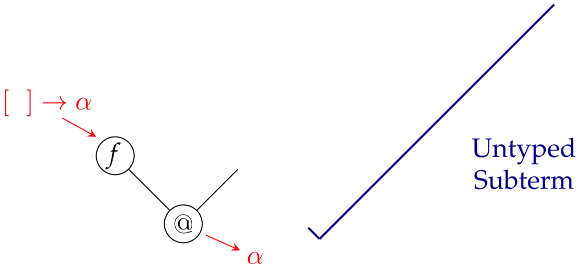
$$\Gamma = f : [[\alpha] \to \alpha]_\omega \text{ (infinite multiplicity)}$$

TRUNCATION (FIGURES)

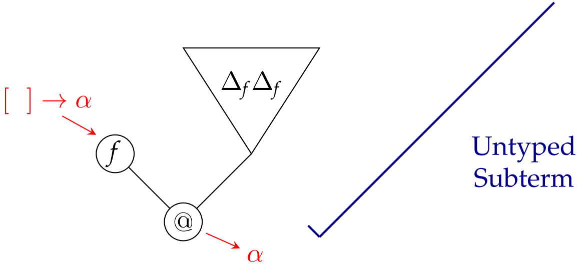
We can use the same derivation frame Π_1^* to type $f(\ldots)$



$\Gamma_1 = f : [[\ ] \to \alpha]$

Truncation (Figures)

$$\Pi^1_1 \quad \rhd \Gamma_1 \vdash f(\Delta_f \Delta_f) : \alpha$$



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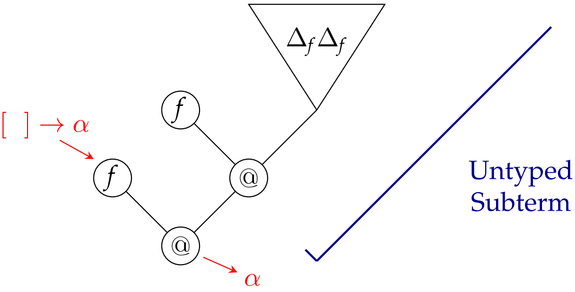
$$\Pi_1^2 \;\; \rhd \Gamma_1 \vdash f(f(\Delta_f \Delta_f)) : \alpha$$



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Truncation (Figures)

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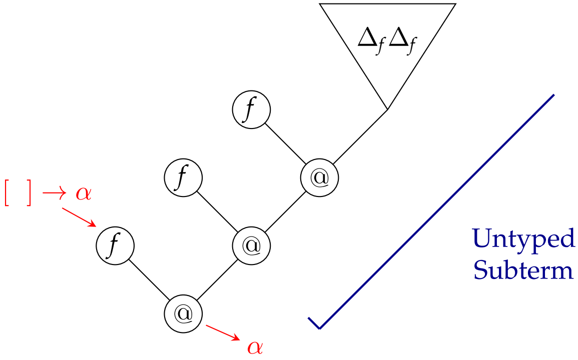

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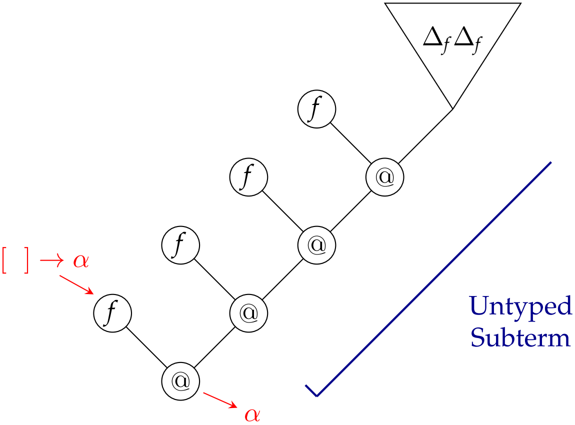

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Truncation (Figures)

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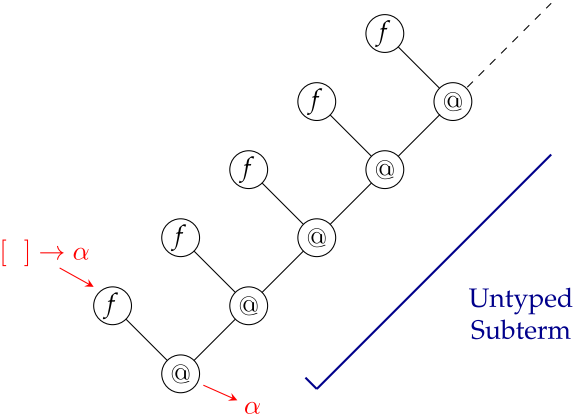

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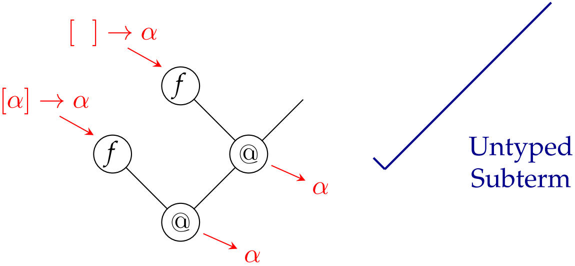
We can use the same derivation frame Π_2^* to type $f(f(\ldots))$



$\Gamma_2 = f : [[\alpha] \to \alpha] + [[\ ] \to \alpha]$

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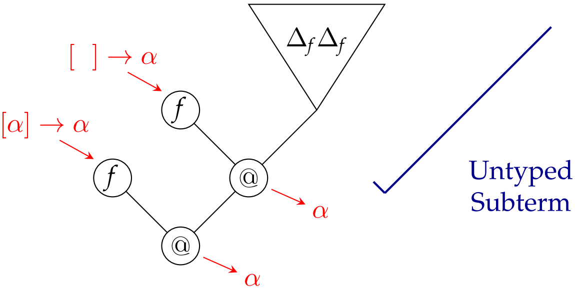

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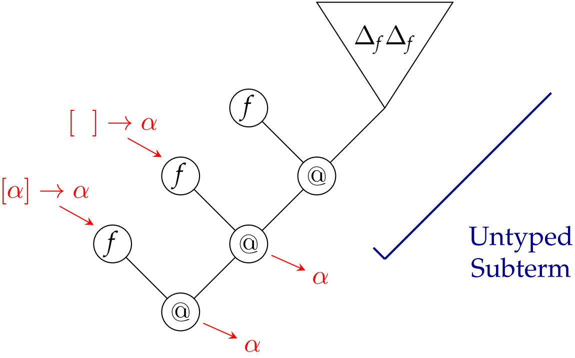

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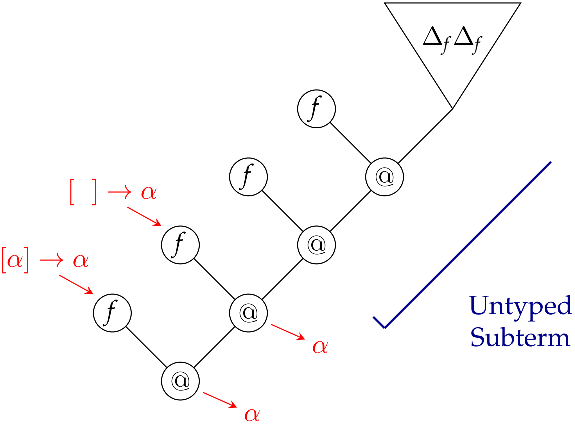

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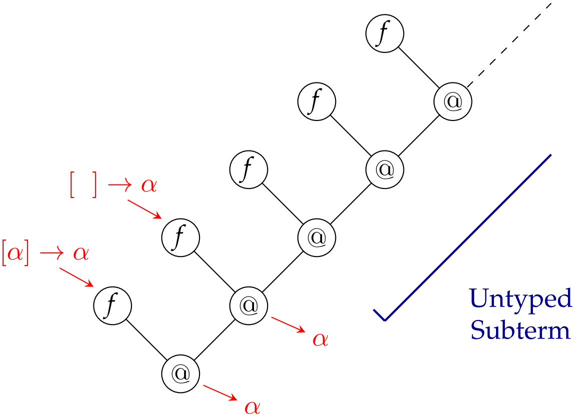

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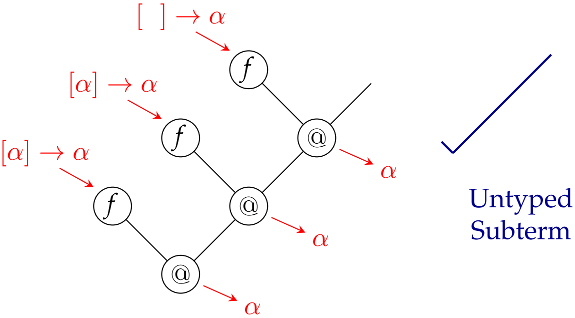

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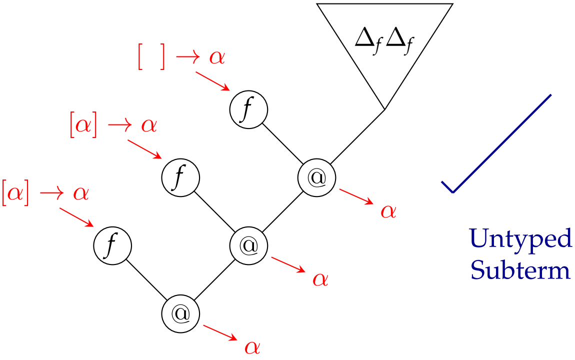

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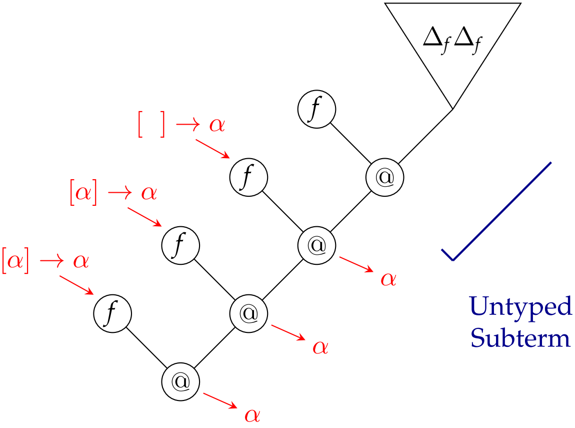

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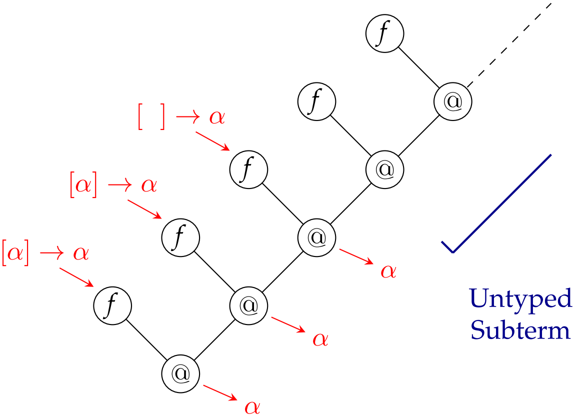

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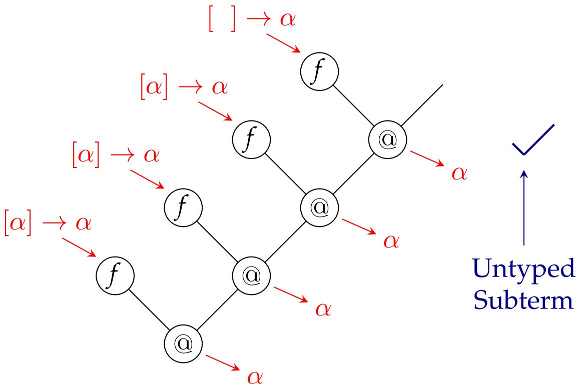

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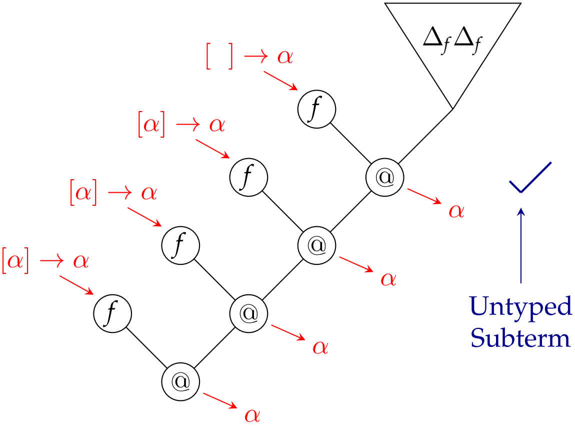

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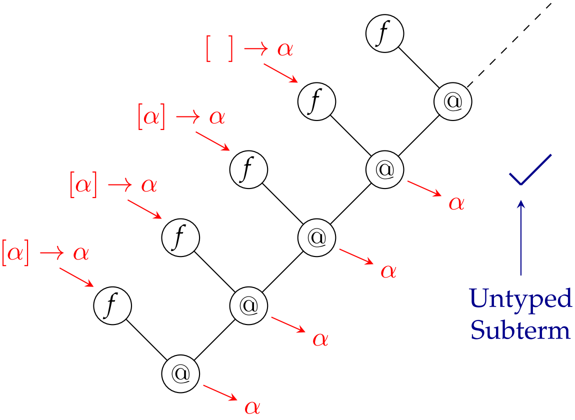

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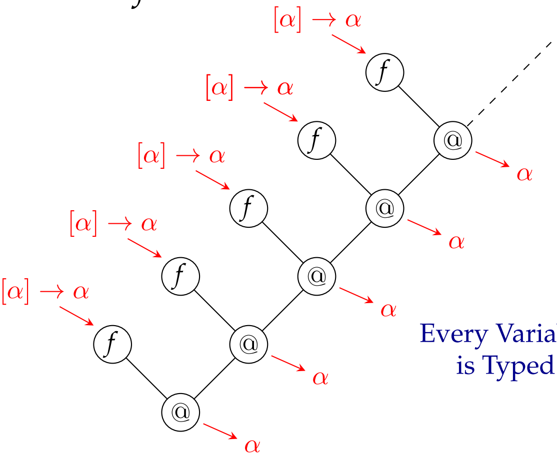

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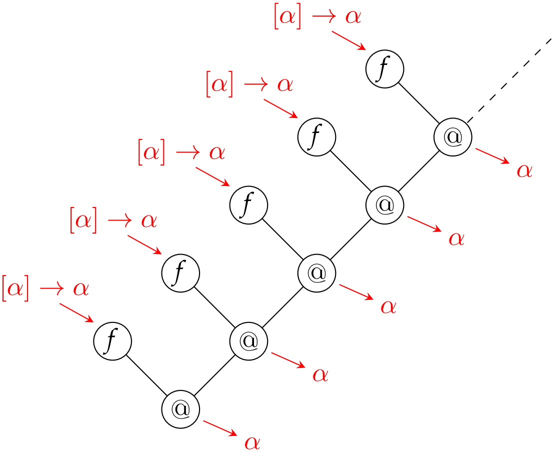

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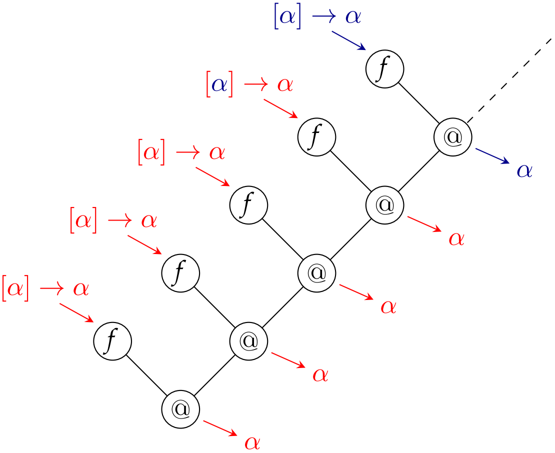

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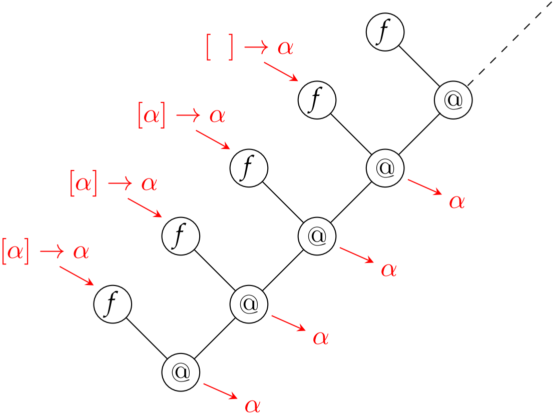

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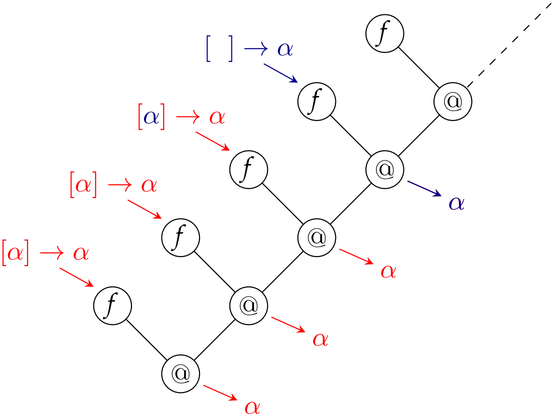

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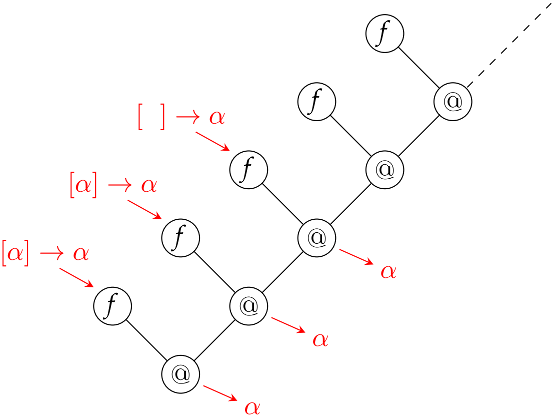

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- Type γ allows to type $\Delta\Delta$. Need for a **validity criterion**.

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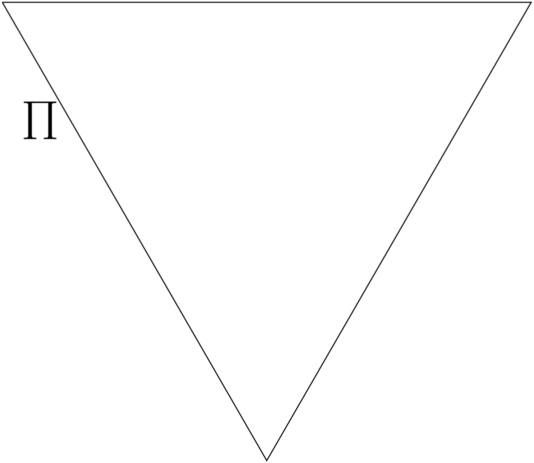
APPROXIMABILITY (FIGURE)



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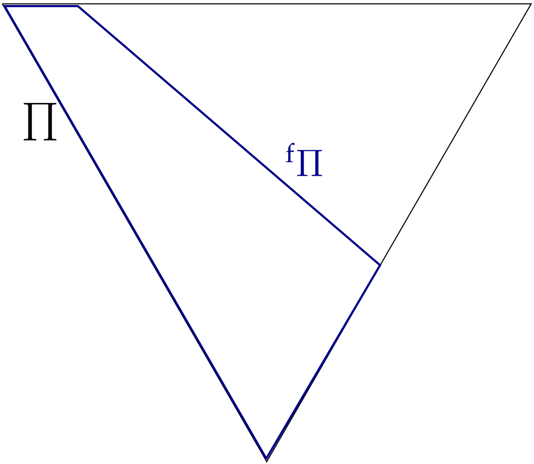

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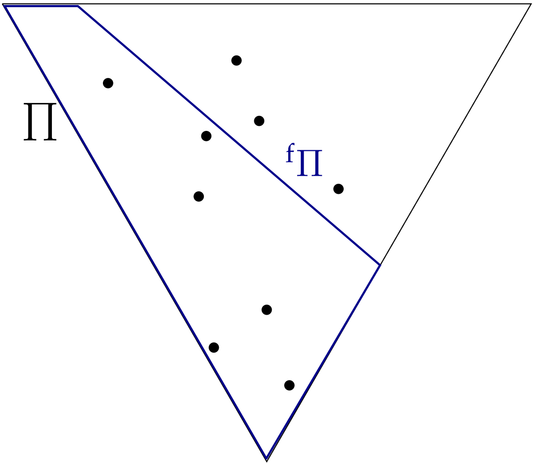

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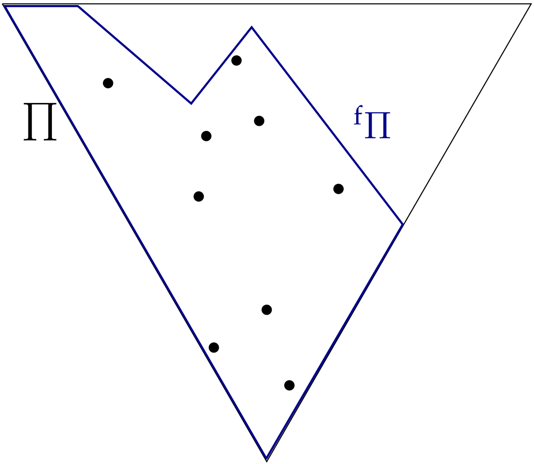
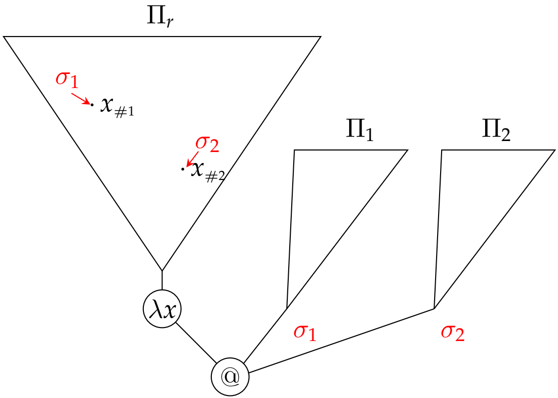


NON-DETERMINISM AND TRUNCATION

$(\lambda x.r)s$



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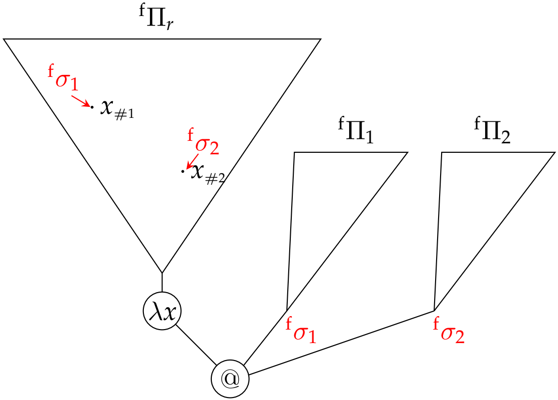

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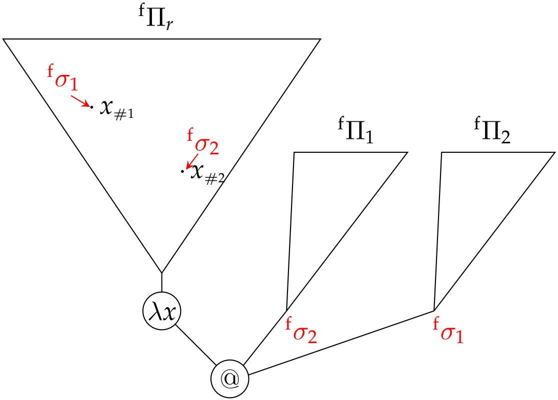

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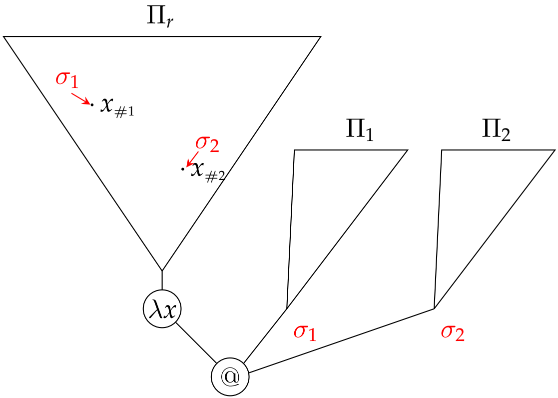

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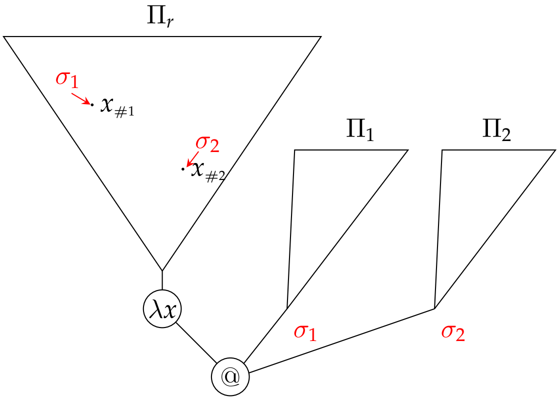

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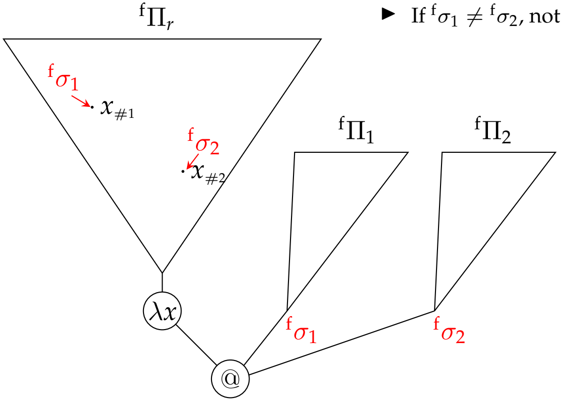

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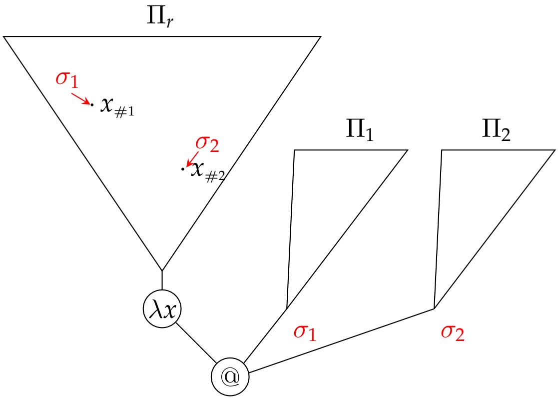

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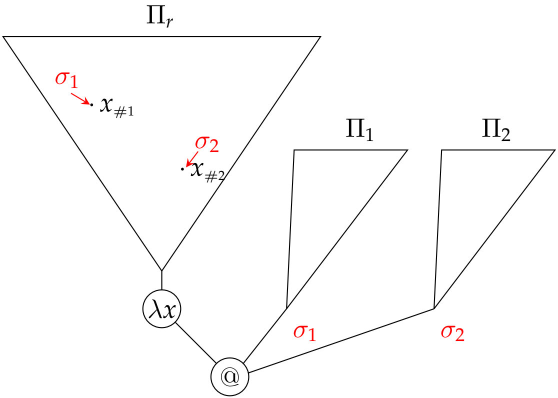

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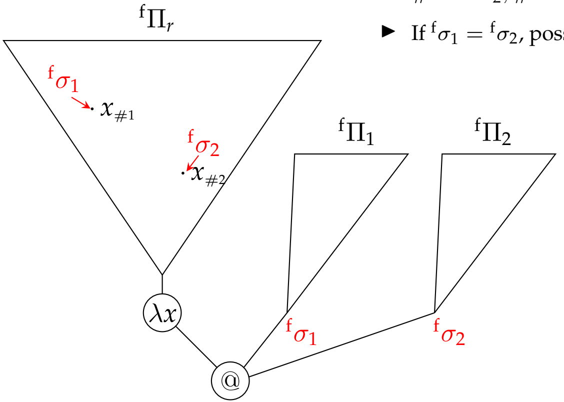

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PLAN

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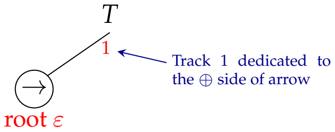

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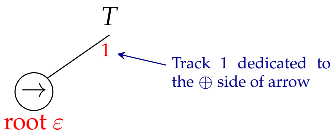

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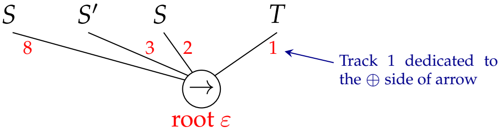

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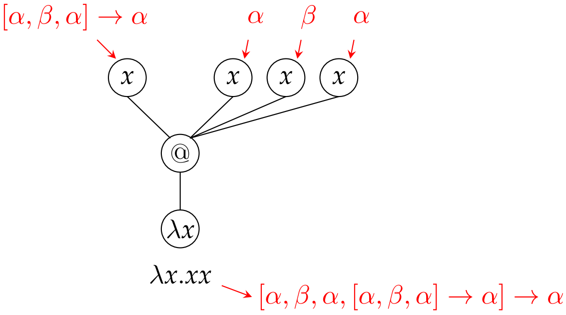

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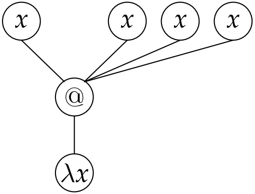

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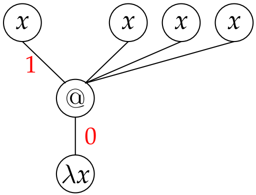

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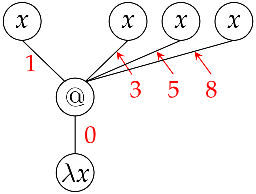

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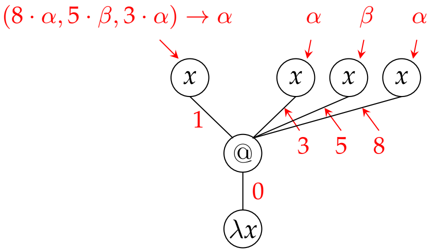

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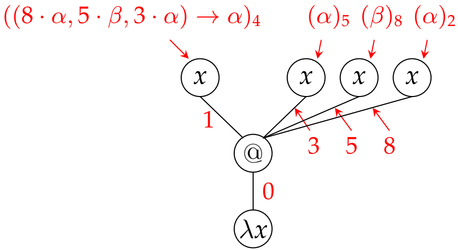

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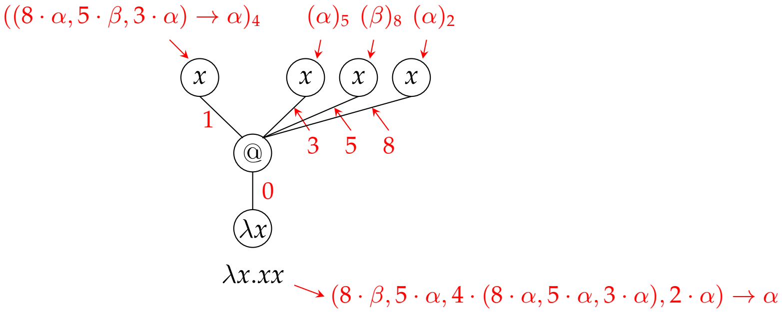

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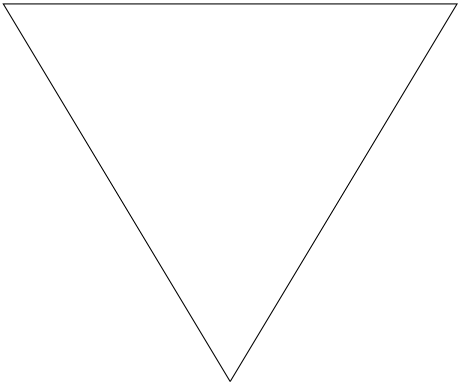
- **Trackability:** every symbol used inside P can be pointed at univocuously. Notion of *biposition*.

BISUPPORT OF A DERIVATION

P



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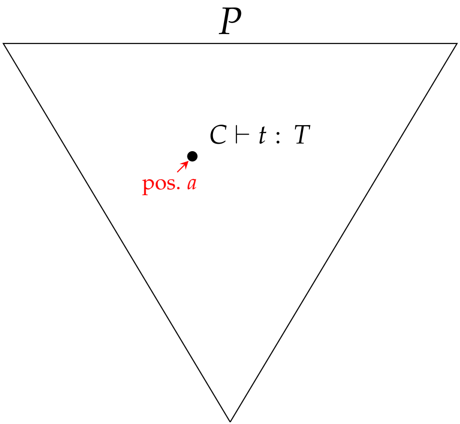

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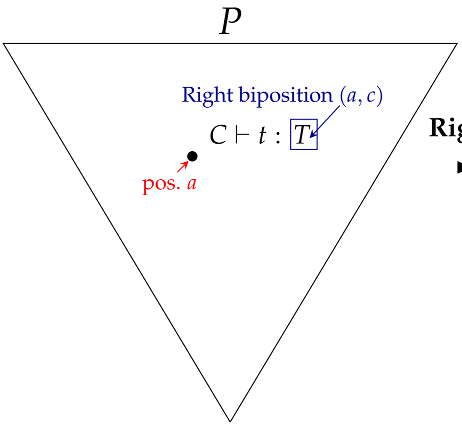

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Right of $\vdash$

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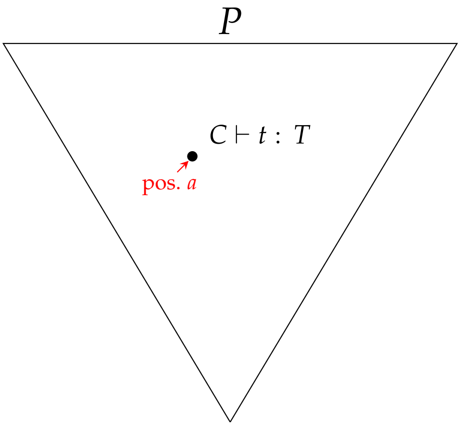

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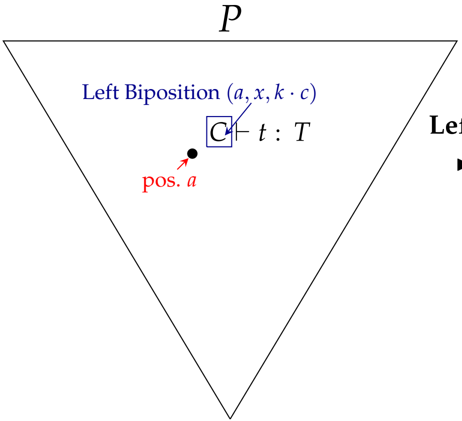

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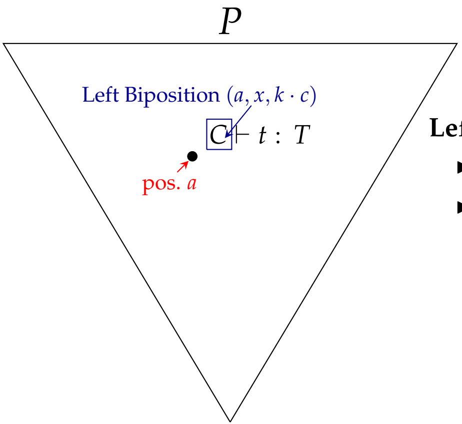

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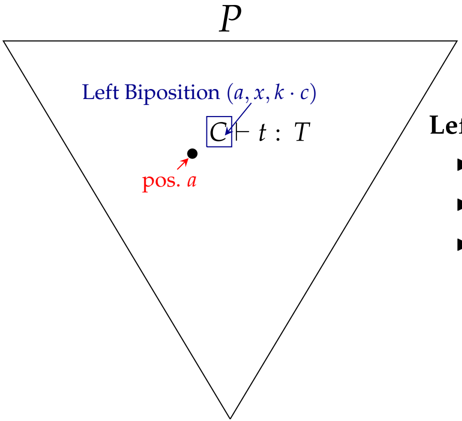

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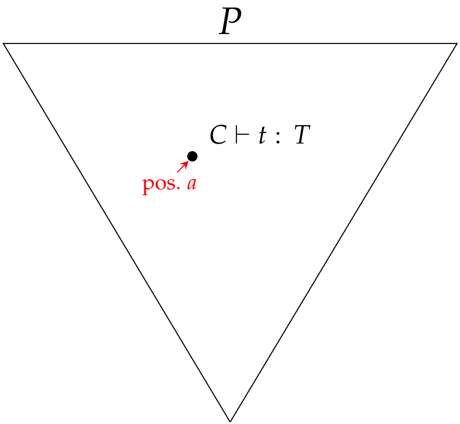

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Bisupport of P: the set of (right or left) bipositions

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- Every symbol inside a rigid derivation P has a **biposition** (a position pointing inside a type nested in a judgment of P).

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- A rigid derivation P is said to be **approximable** if for all finite part B of P, there is a finite approximation ${}^{\mathrm{f}}P \leqslant P$ s.t. ${}^{\mathrm{f}}P$ contains B.

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Argument 4: Subject expansion property holds for s.c.r.s. (assuming approximability only).

PLAN

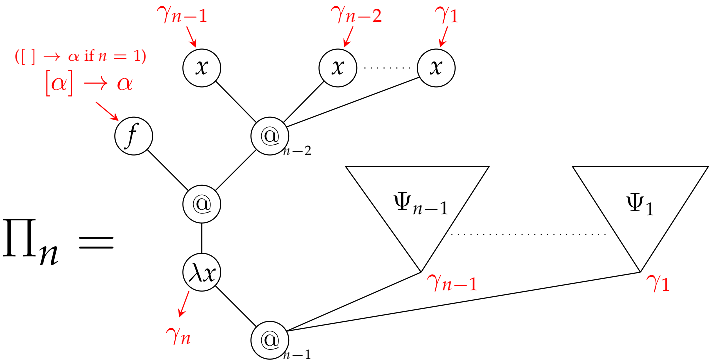
FUTURE WORK?

- Representation Theorem: every $\mathscr{M}$-derivation is the collapse of a $\mathtt{S}$-derivation (already done, HOR 2016).
- Can we reformulate approximability ?
- Can infinitary Strong Normalization be characterized ?
- Is every term typable in $\mathtt{S}$ (without approximability) ?
  Yes ! $\mathtt{S}$ is completely unsound (difficult because of relevance).
- *Categorical Adaptation* of this framework (ongoing work with D. Mazza and L. Pellisier).

QUESTIONS

Thank you for your attention !

Expanded Derivations



- Ψ_n is the left subderivation.
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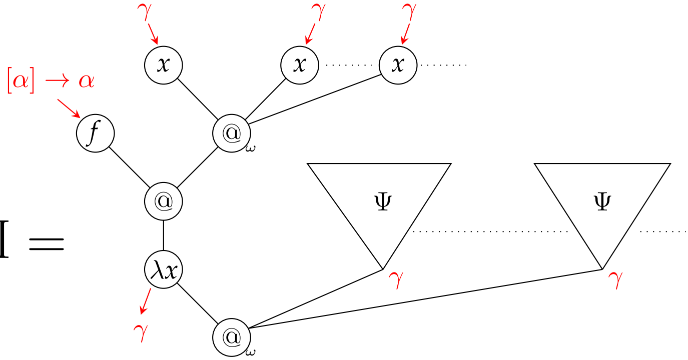
$\Pi =$ 

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- To obtain System $\mathscr{M}$, take the rules of $\mathscr{M}_0$ coinductively (with those types and multiset types).