Some Uses of Infinitary Intersection Types as Sequences

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- Another use of denotations: **equating** or **separating programs** *i.e.* two states that have different denotations cannot be instances of the same program.

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- When a type system enjoys **subject reduction** and **expansion**, types are execution invariants (and they usually provide us with models of λ-calculus).

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- Some programs are non terminating but **productive**.
- Many possible definitions or variants of sound non termination
  Klop and alii[95], Endrullis,Polonsky and alii[15]

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- The **collapse** of System $\mathtt{S}$ on System $\mathscr{R}$ is **surjective**.
  Every multiset based derivation is the collapse of a sequence based derivation. No loss of expressivity while resorting to $\mathtt{S}$.

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- *Example:* usually, xx cannot be typed in STS, but xx can be typed in ITS: if x is assigned $A \wedge (A \to B)$, then $xx : B$ is derivable.

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- In-between possibility: **rigidity**.
  Paradigm: **sequences**

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  - $(2 \cdot a,\ 3 \cdot b,\ 8 \cdot a) \uplus (3 \cdot b,\ 9 \cdot c)$ **not** defined (**incompatibility**).

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  - Inductively, a term is WN if it is HN and all the head arguments are themselves WN.

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- A term is **head-normalizing (HN)** if it can be reduced to a HNF (in a finite number of steps)

- Coinductively, a term is **hereditary head-normalizing (HHN)** if it can be reduced to a HNF and all the head arguments are themselves HHN.

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- Can a coinductive ITS characterize the set of HHN terms?

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- Present the key notions of **truncations** and **approximability** (meant to avoid *irrelevant* derivations).
- Understand why **commutative intersection** is **unfit** to express those key notions.
- Present the coinductive type assignment system $\mathtt{S}$: intersection types are **sequences** of types, instead of *sets* of types (idempotent intersection fw.) or *multisets* of types (regular non-idempotent fw.).

PLAN

TYPING RULES OF $\mathscr{R}_0$ (GARDNER/DE CARVALHO)

Types (τ, σ_i): $\tau, \sigma_i := \ o \in \mathscr{O} \mid [\sigma_i]_{i \in I} \to \tau.$

Context (Γ, Δ): assign *intersection* types to variables.

$$\frac{}{x : [\tau] \vdash x : \tau}\ \texttt{ax} \qquad \frac{\Gamma, x : [\sigma_i]_{i \in I} \vdash t : \tau}{\Gamma \vdash \lambda x.t : [\sigma_i]_{i \in I} \to \tau}\ \texttt{abs}$$

$$\frac{\Gamma \vdash t : [\sigma_i]_{i \in I} \to \tau \qquad (\Delta_i \vdash u : \sigma_i)_{i \in I}}{\Gamma +_{i \in I} \Delta_i \vdash t\,u : \tau}\ \texttt{app}$$

Examples:

$$\dfrac{\dfrac{}{x : [\tau] \vdash x : \tau}\ \texttt{ax}}{\vdash \lambda x.x : [\tau] \to \tau}\ \texttt{abs} \qquad \dfrac{\dfrac{}{x : [\tau] \vdash x : \tau}\ \texttt{ax}}{x : [\tau] \vdash \lambda y.x : [\,] \to \tau}\ \texttt{abs}$$

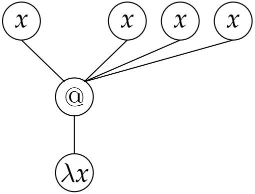
ALTERNATIVE PRESENTATION

Standard presentation

$$\dfrac{\dfrac{}{x : [[\alpha,\beta,\alpha] \to \alpha] \vdash x : [\alpha,\beta,\alpha] \to \alpha}\ \text{ax} \quad \dfrac{}{x : [\alpha] \vdash x : \alpha}\ \text{ax} \quad \dfrac{}{x : [\beta] \vdash x : \beta}\ \text{ax} \quad \dfrac{}{x : [\alpha] \vdash x : \alpha}}{\dfrac{x : [\alpha,\beta,\alpha,[\alpha,\beta,\alpha] \to \alpha] \vdash xx : \alpha}{\vdash \lambda x.xx : [\alpha,\beta,\alpha,[\alpha,\beta,\alpha] \to \alpha] \to \alpha}\ \text{abs}}$$

ALTERNATIVE PRESENTATION

Alternative presentation



$\lambda x.xx$

- Indicate the arity of application rules.

Alternative Presentation

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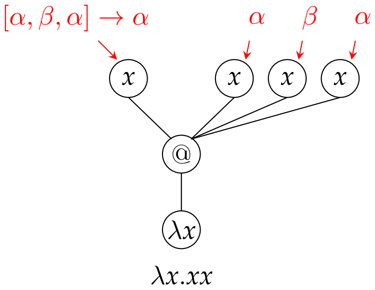

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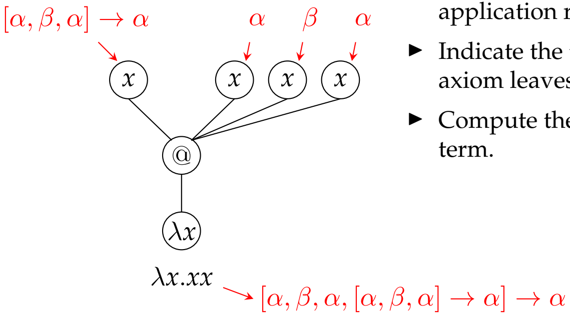

- Indicate the arity of application rules.
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- Compute the type of the term.

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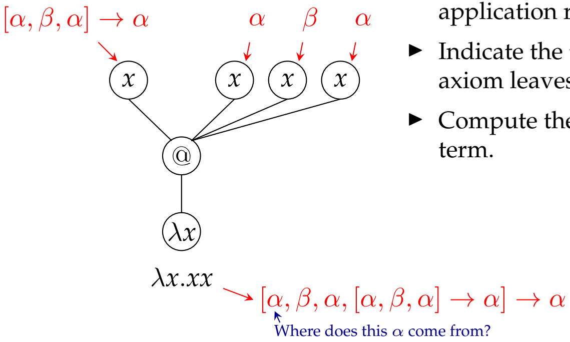

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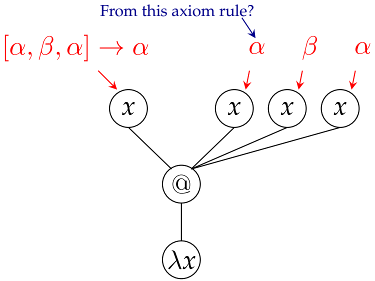

$\lambda x.xx$ $\to$ $[\alpha, \beta, \alpha, [\alpha, \beta, \alpha] \to \alpha] \to \alpha$

Where does this α come from?

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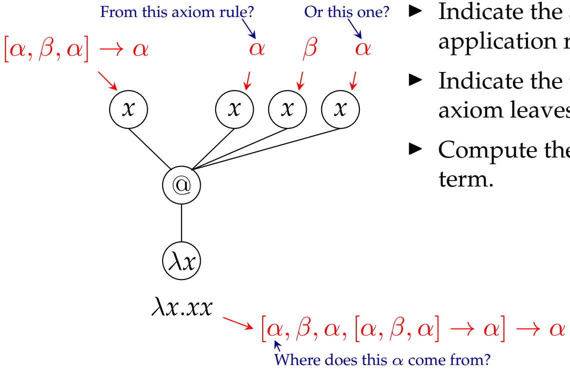

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Axiom leaves
typing x inside Π_r

$$\dfrac{\dfrac{\begin{array}{c}\Pi_r \\ \vdots \quad \left(\dfrac{}{x : [\sigma_i] \vdash x : \sigma_i}\,\mathtt{ax}\right)_{i \in I} \\ \vdots \\ \Gamma, x : [\sigma_i]_{i \in I} \vdash r : \tau\end{array}}{\Gamma \vdash \lambda x.r : [\sigma_i]_{i \in I} \to \tau}\,\mathtt{abs} \qquad \left(\begin{array}{c}\Pi_i \\ \vdots \\ \Delta_i \vdash s : \sigma_i\end{array}\right)^{i \in I}}{\Gamma + \sum_{i \in I} \Delta_i \vdash (\lambda x.r)s : \tau}\,\mathtt{app}$$

SUBJECT REDUCTION PROPERTY FOR $\mathscr{M}_0$

If $\Pi \rhd \Gamma \vdash t : \tau$ and $t \to t'$, then $\exists \Pi' \rhd \Gamma \vdash t' : \tau$

$$(\lambda x.r)s \to r[s/x]$$

$$\dfrac{\dfrac{\begin{array}{c}\Pi_r \\ \left(\dfrac{}{x : [\sigma_i] \vdash x : \boxed{\sigma_i}}\,\mathtt{ax}\right)_{i\in I} \\ \vdots \\ \Gamma, x : [\sigma_i]_{i\in I} \vdash r \quad : \tau\end{array}}{\Gamma \vdash \lambda x.r : [\sigma_i]_{i\in I} \to \tau}\,\mathtt{abs} \qquad \left(\begin{array}{c}\Pi_i\\ \vdots \\ \Delta_i \vdash s : \boxed{\sigma_i}\end{array}\right)^{i\in I}}{\Gamma + \sum_{i\in I}\Delta_i \vdash (\lambda x.r)s : \tau}\,\mathtt{app}$$

Subject Reduction Property for $\mathscr{M}_0$

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$$(\lambda x.r)s \to r[s/x]$$

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"association"

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$$(\lambda x.r)s \to r[s/x]$$

$$\dfrac{\dfrac{\begin{array}{c}\Pi_r\\ \left(\dfrac{}{\cancel{x : [\sigma_i] \vdash x} : \boxed{\sigma_i}}\,\mathtt{ax}\right)_{i\in I}\\ \vdots\\ \Gamma, x : [\sigma_i]_{i\in I} \vdash r \quad : \tau\end{array}}{\cancel{\Gamma \vdash \lambda x.r : [\sigma_i]_{i\in I} \to \tau}}\,\mathtt{abs} \qquad \text{“association”} \qquad \left(\begin{array}{c}\Pi_i\\ \vdots\\ \Delta_i \vdash s : \boxed{\sigma_i}\end{array}\right)^{i\in I}}{\cancel{\Gamma + \sum_{i\in I}\Delta_i \vdash (\lambda x.r)s : \tau}}\,\mathtt{app}$$

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$$\begin{array}{c}
\begin{array}{cc}
\Pi_r & \left(\begin{array}{c} \Pi_i \\ \vdots \\ \Delta_i \vdash s : \sigma_i \end{array} \right)^{i \in I} \\
\vdots & \\
\end{array} \\
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\end{array}$$

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$$(\lambda x.r)s \longrightarrow r[s/x]$$

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Vocabulary:
We say each **association** (between x-axiom leaves and arg-derivations) yields a **derivation reduct** Π' typing $r[s/x]$.

SUBJECT REDUCTION PROPERTY FOR $\mathscr{M}_0$

If $\Pi \rhd \Gamma \vdash t : \tau$ and $t \to t'$, then $\exists \Pi' \rhd \Gamma \vdash t' : \tau$

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Observation:
If a type σ occurs several times in $[\sigma_i]_{i \in I}$, there can be several associations, each one yielding a possibly different derivation reducts Π'.

NORMALIZABILITY RESULTS

Proposition

A term is HN iff it is typable in $\mathscr{R}_0$.

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A judgement $\Gamma \vdash t : \tau$ is **unforgetful** if there is no negative occurrence of $[\,]$ in Γ and no positive occurrence of $[\,]$ in τ.

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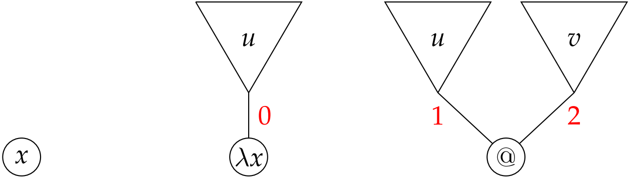
Definition

A judgement $\Gamma \vdash t : \tau$ is **unforgetful** if there is no negative occurrence of $[\,]$ in Γ and no positive occurrence of $[\,]$ in τ.

- $[\,]$ occurs negatively in $[\,] \to \tau$
- If $[\,]$ occurs negatively in σ_2 then $[\,]$ occurs positively in $[\sigma_1, \sigma_2, \sigma_3] \to \tau$ and so on.

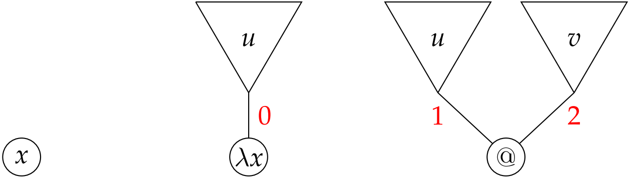
PLAN

∞-TERMS



Variable x **Abstraction** $\lambda x.u$ **Application** $u\,v$

∞-TERMS



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- **Position**: finite sequence in $\{0, 1, 2\}^*$, *e.g.* $0 \cdot 0 \cdot 2 \cdot 1 \cdot 2$.

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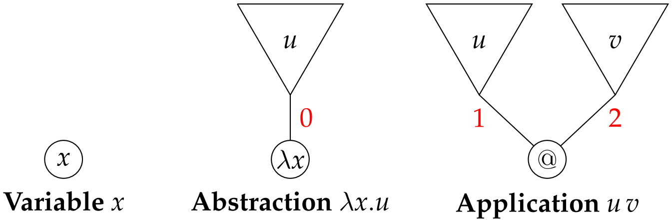

- **Position**: finite sequence in $\{0,1,2\}^*$, *e.g.* $0 \cdot 0 \cdot 2 \cdot 1 \cdot 2$.
- **Applicative Depth (a.d.):** number of $\nearrow$-edges *e.g.*

$$\mathtt{ad}(1 \cdot 2 \cdot 2 \cdot 0 \cdot 2 \cdot 1 \cdot 2) = 4$$

001-TERMS

Λ^{001}: the set of ∞-terms t s.t.:

$$\texttt{br} \text{ is an infinite branch of } t \Rightarrow \texttt{ad}(\texttt{br}) = \infty.$$

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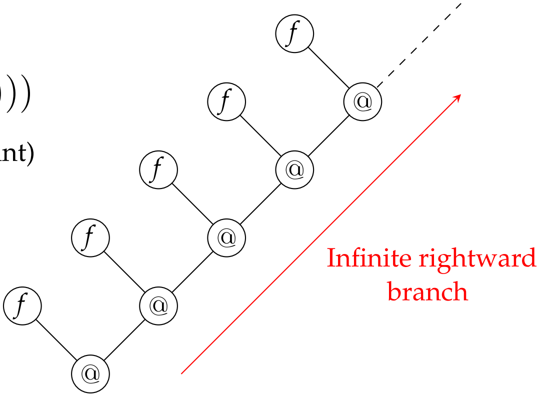
$$f^{\omega} := f(f(f(\ldots)))$$

i.e. $f^{\omega} = f(f^{\omega})$ (fixpoint)



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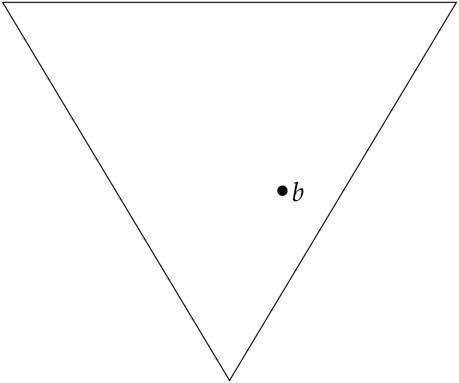

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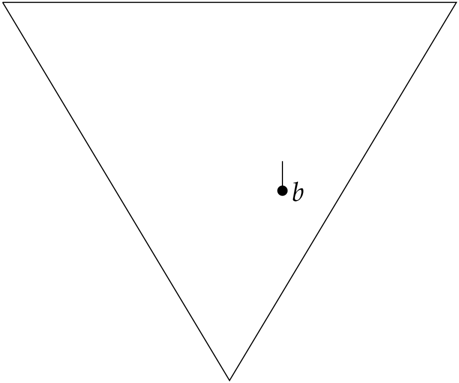

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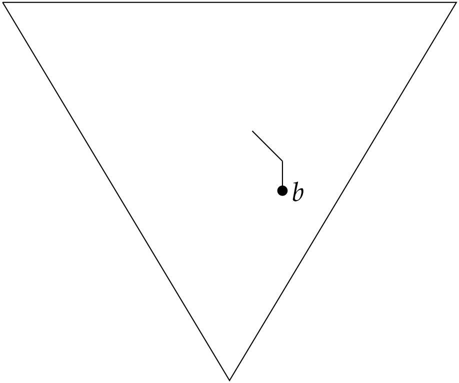

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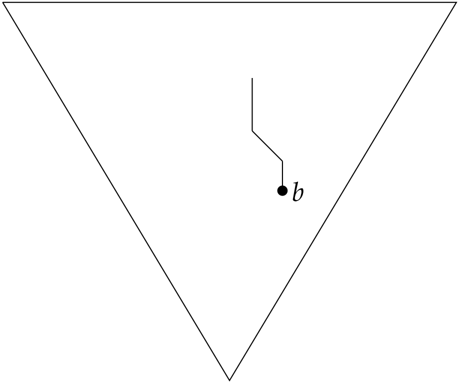

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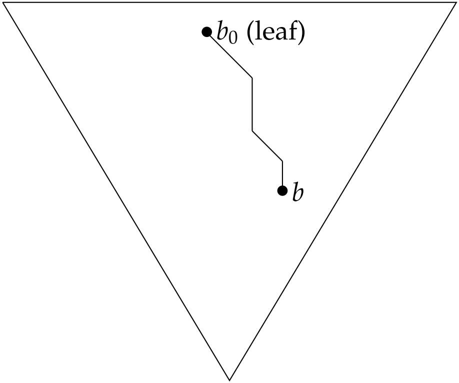


- Start from $b \in \texttt{supp}(t)$
- Move $\uparrow$ or $\nwarrow$
  a.d. does not increase
- A leaf b_0 must be reached

STRONG CONVERGENCE

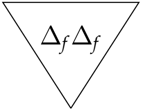
Definition

A reduction sequence $t_0 \xrightarrow{b_0} t_1 \xrightarrow{b_1} t_2 \xrightarrow{b_2} \ldots \xrightarrow{b_{n-1}} t_n \xrightarrow{b_n} \ldots$ is **strongly converging** if it is of finite length or if $\lim \mathtt{ad}(b_n) = \infty$.

Strong Convergence

$$\Delta_f := \lambda x.f(xx) \qquad\qquad \Delta_f\Delta_f\text{: "Curry"}$$

$$\Delta_f\Delta_f \to f(\Delta_f\Delta_f) \to f^2(\Delta_f\Delta_f) \to f^3(\Delta_f\Delta_f) \to f^4(\Delta_f\Delta_f) \to \ldots \to^\infty f^\omega$$



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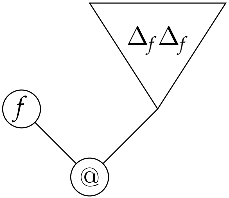

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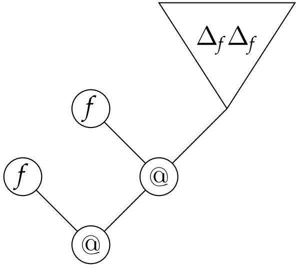

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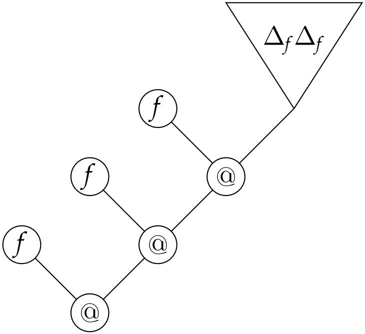

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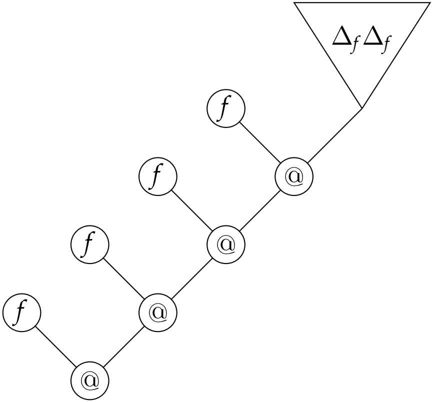

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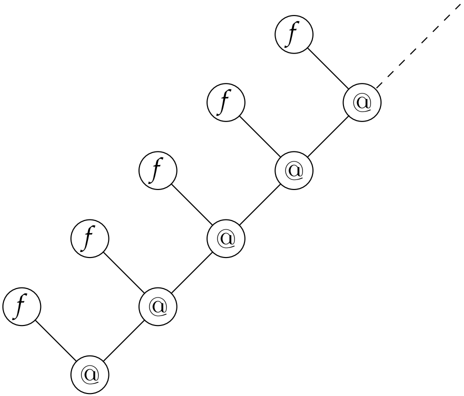

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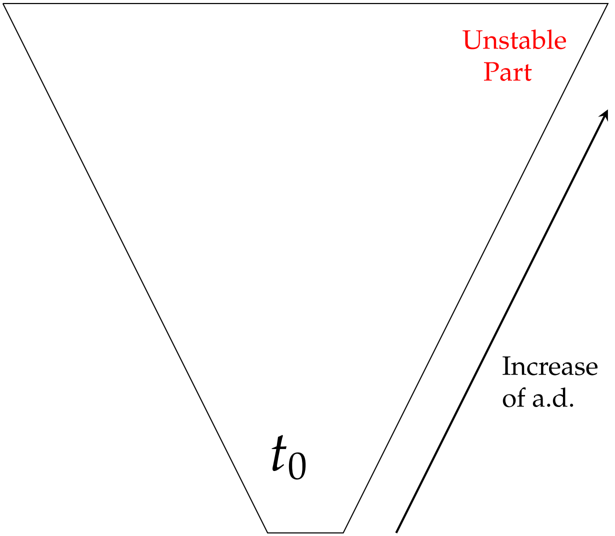

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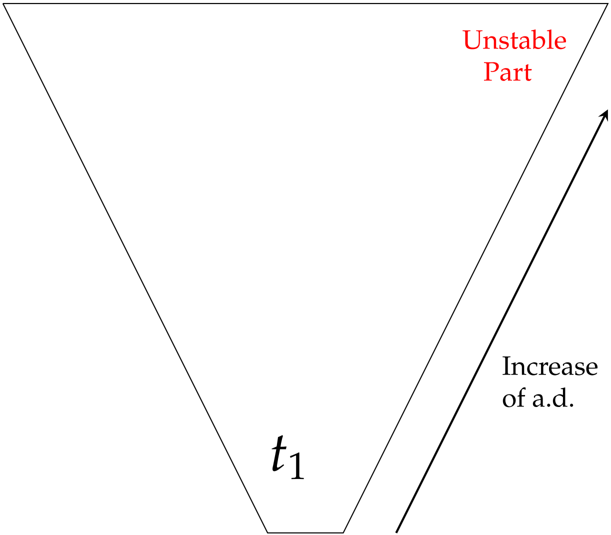

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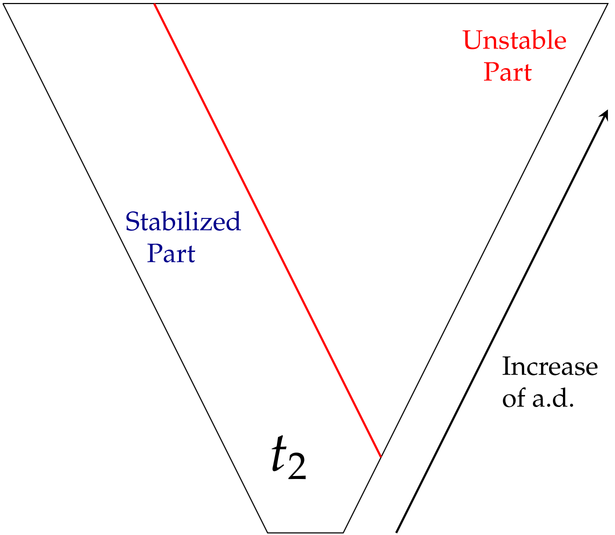

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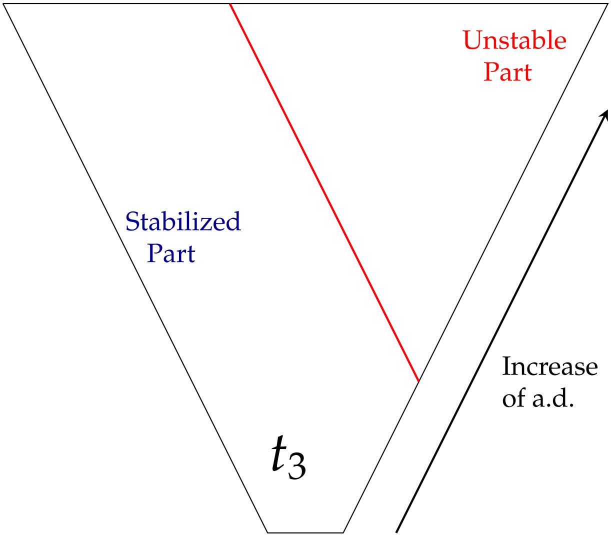

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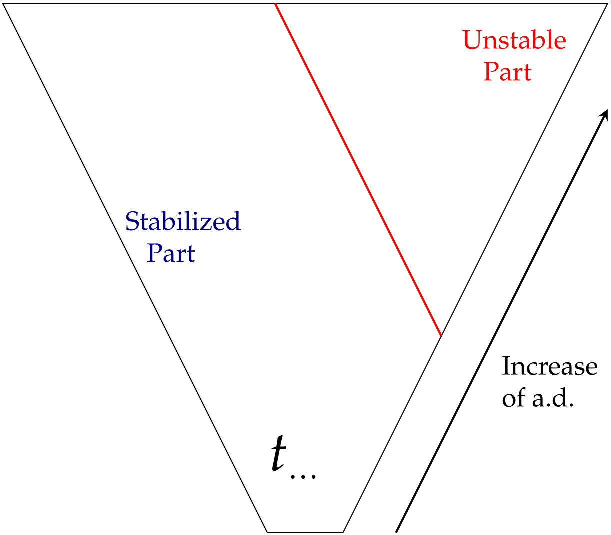

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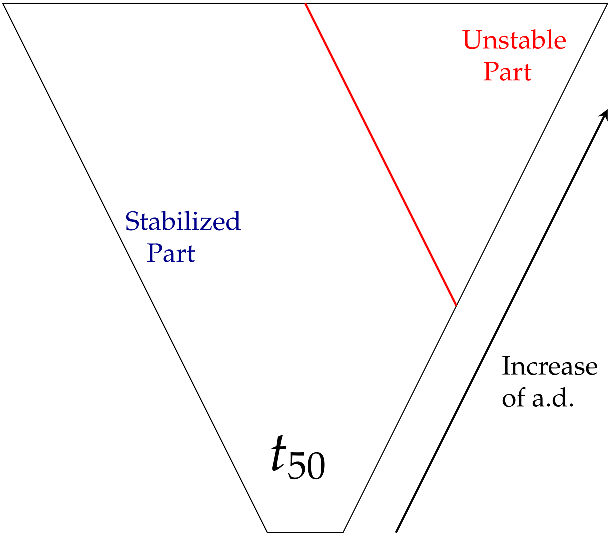

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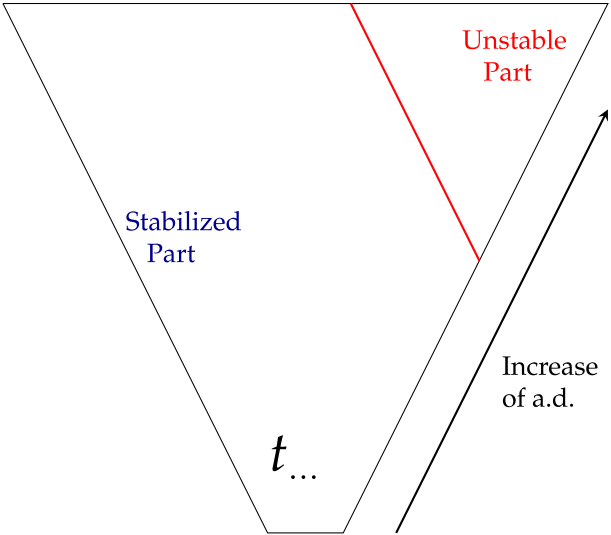

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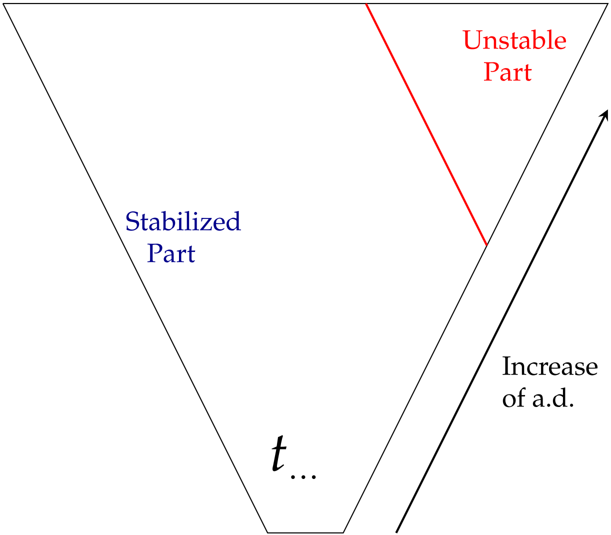

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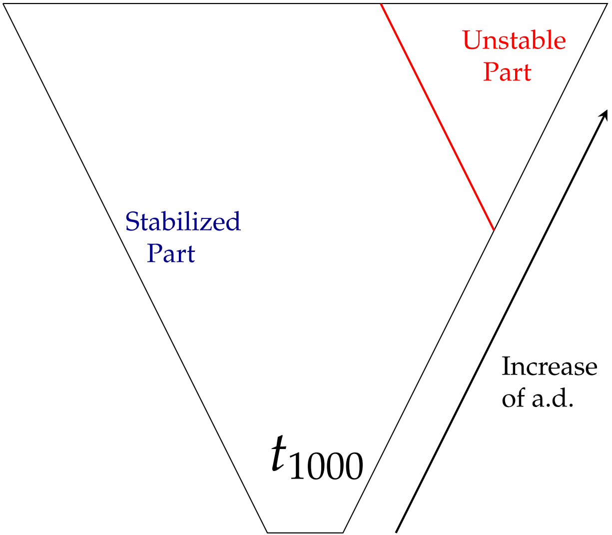

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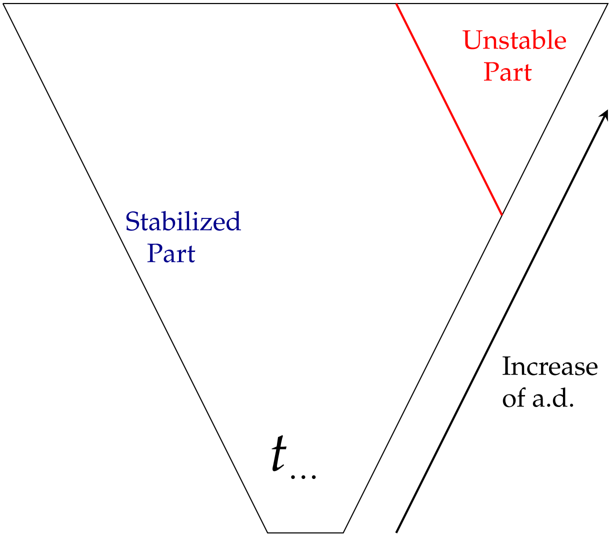

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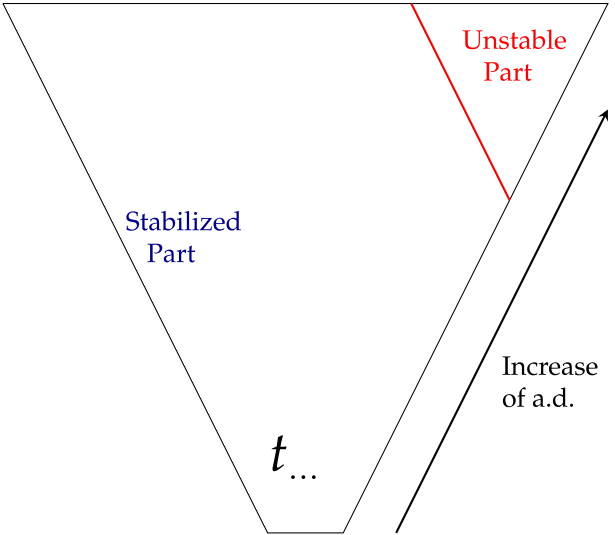

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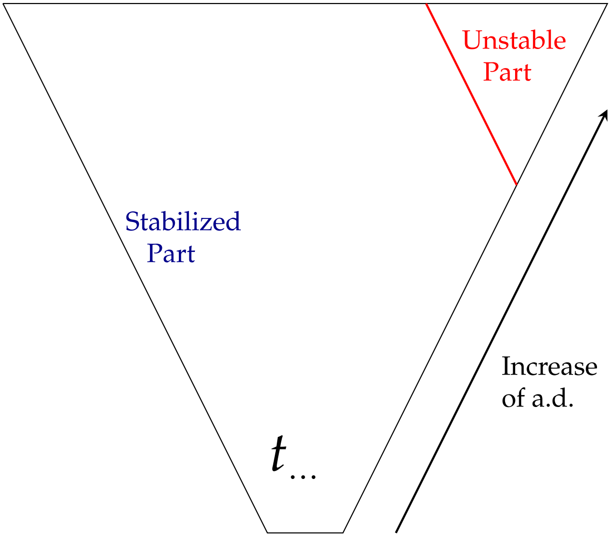

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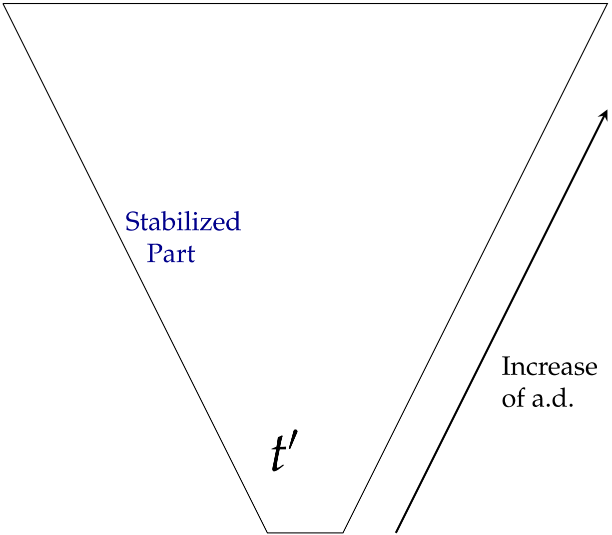

STRONG CONVERGENCE

Conclusion

Strong Convergence

Conclusion

A **strongly converging reduction sequence (s.c.r.s)** allows us to define its **limit**.

INFINITARY NORMALIZATION

- The notions of redex and head-normalizability do not change.

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Definition (Infinitary WN)

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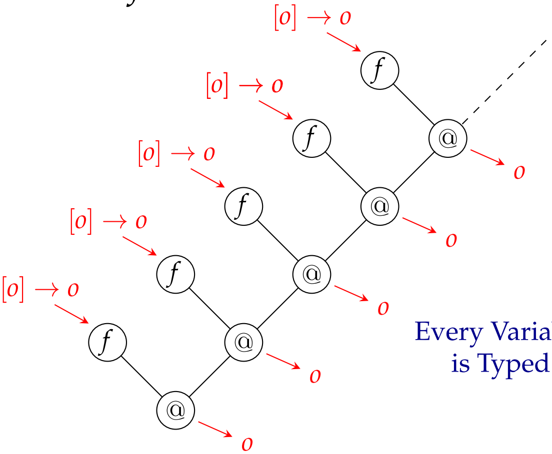
- Thus, a (finite) term is HHN iff it is 001-WN.

Plan

Truncation (Figures)

$$\Pi' \rhd \Gamma \vdash f^\omega : o$$

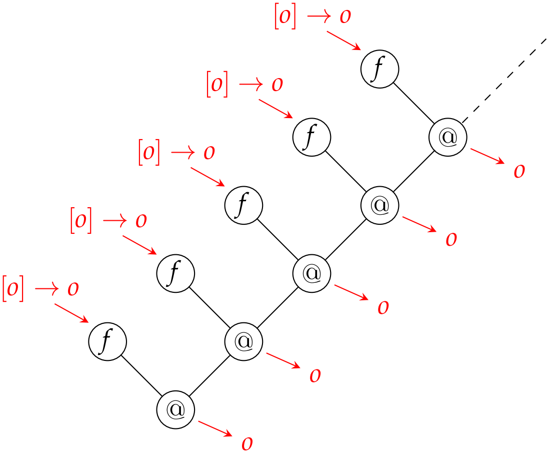


Every Variable
is Typed

$$\Gamma = f : [[o] \to o]_\omega \text{ (infinite multiplicity)}$$

TRUNCATION (FIGURES)

Π' can be **truncated** into Π'_4:



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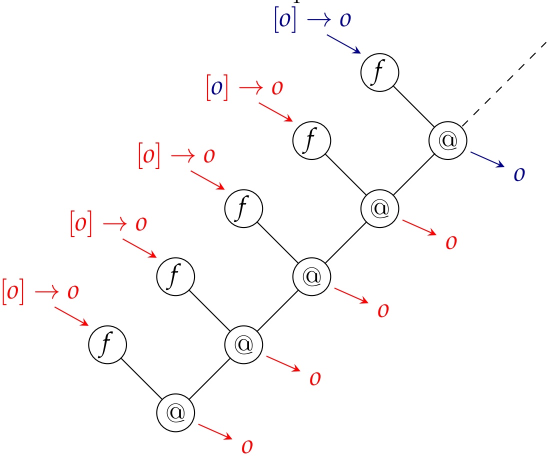

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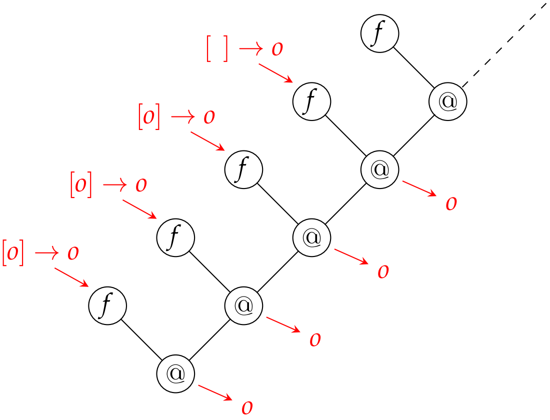

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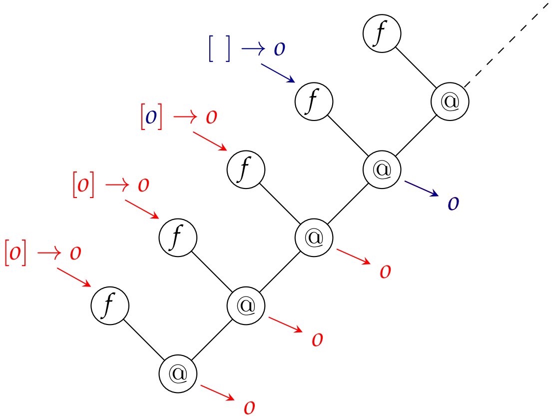
Π' can be **truncated** into Π'_3:



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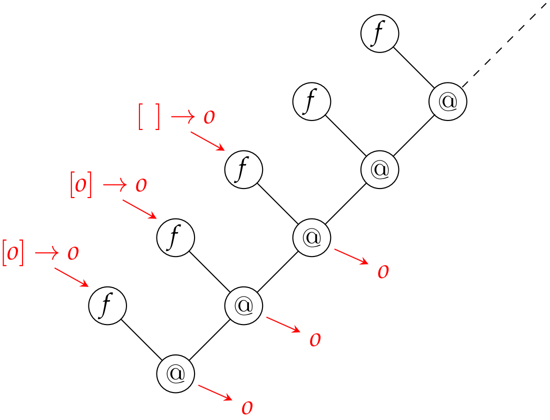
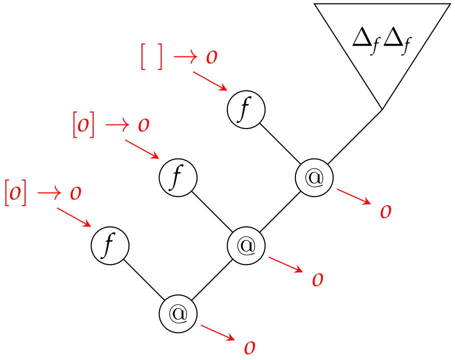


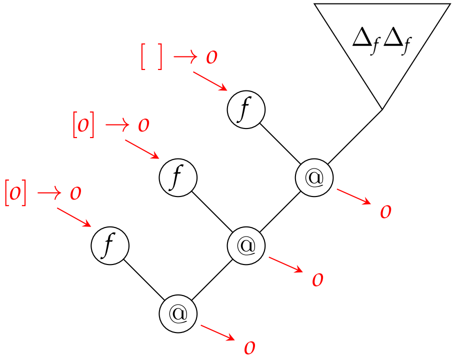
Truncation (Figures)

f^ω may be replaced by $f^3(\Delta_f\Delta_f)$ in Π_3',
yielding Π_3^3 :



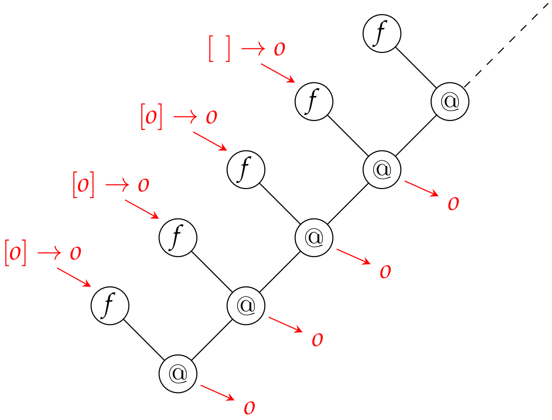
TRUNCATION (FIGURES)

Π_3^3 may be expanded 3 times,
yielding $\Pi_3 \rhd \Delta_f \Delta_f$:



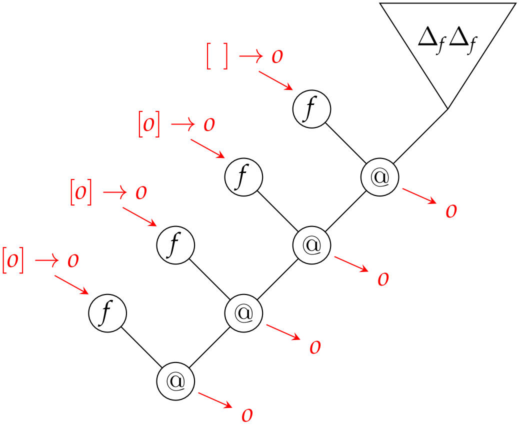
TRUNCATION (FIGURES)

Back to Π'_4, level 4 truncation of Π' :



TRUNCATION (FIGURES)

f^ω may be replaced by $f^4(\Delta_f\Delta_f)$ in Π_3',
yielding Π_4^4 :



TRUNCATION (FIGURES)

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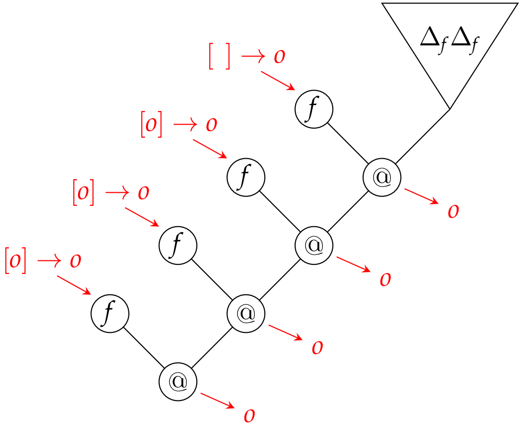

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- Take the join of all the Π_n (while $n \to \infty$): this defines Π, the desired expansion of Π'.

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- Type γ allows to type $\Delta\Delta$. Need for a **validity criterion**.

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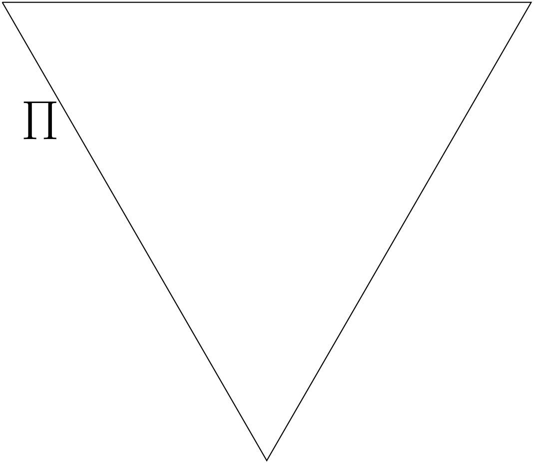

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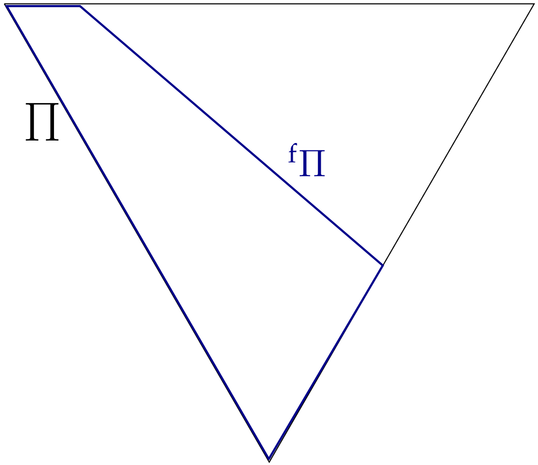

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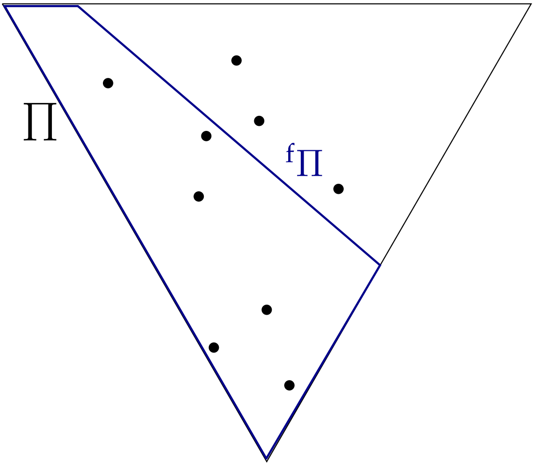

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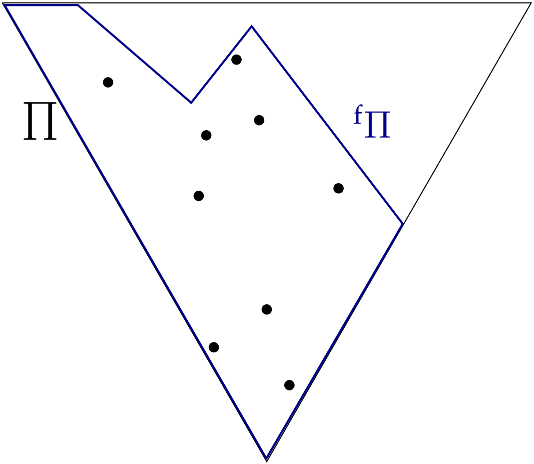
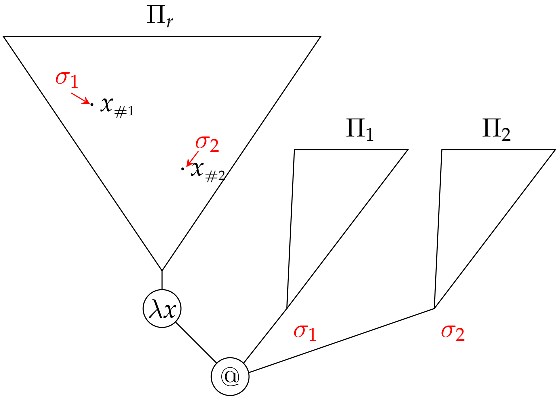


NON-DETERMINISM AND TRUNCATION

$$(\lambda x.r)s$$



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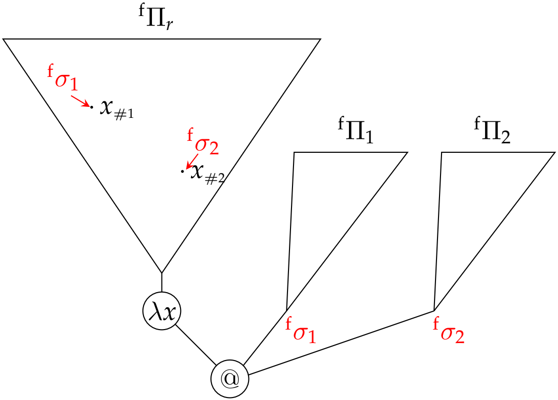

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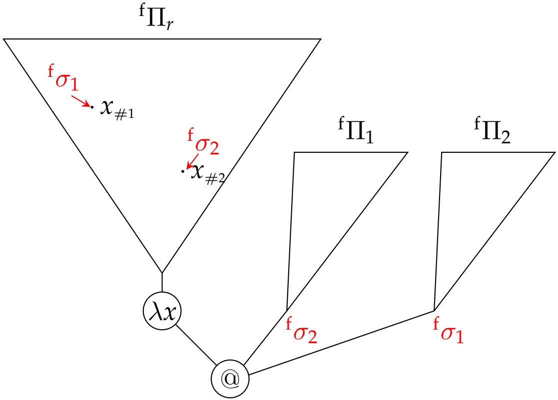

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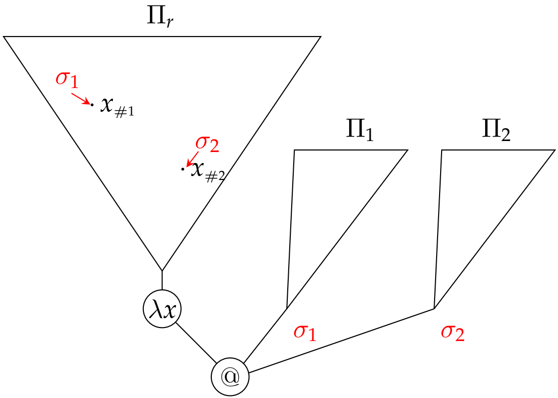

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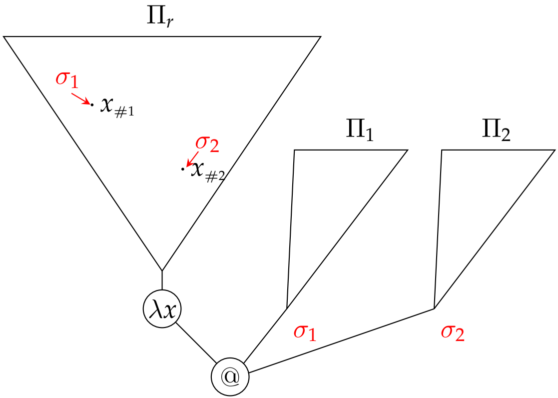

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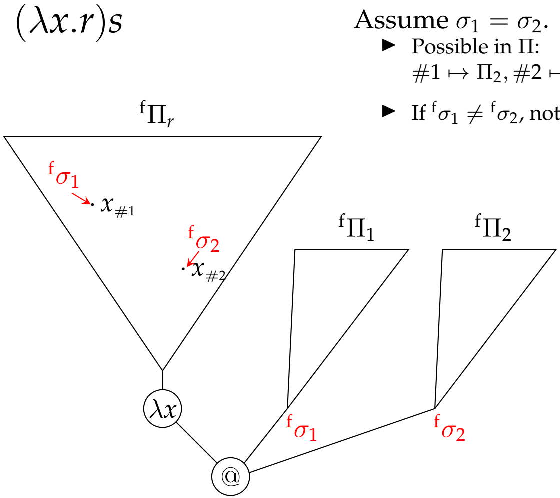

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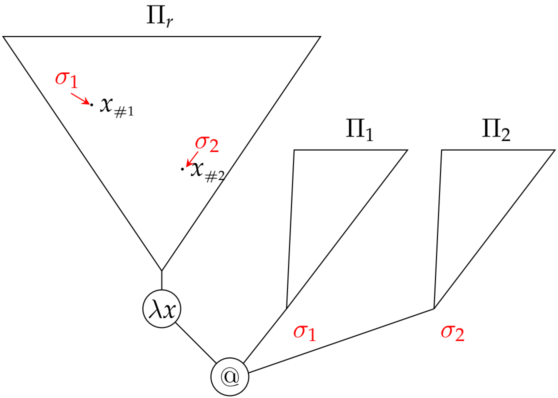

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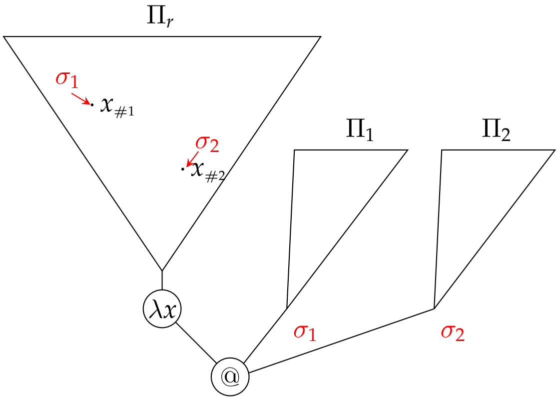

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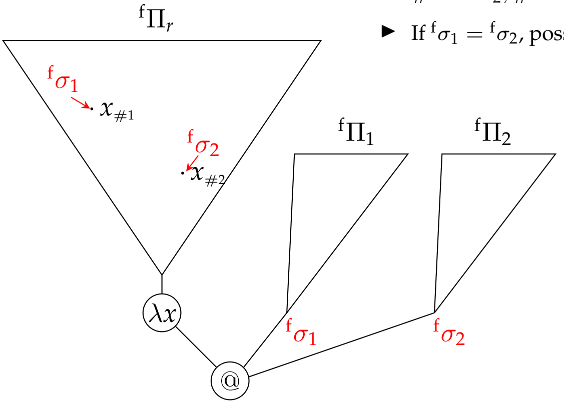

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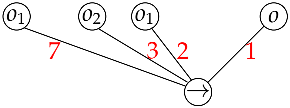
- *Example:* $(7 \cdot o_1, 3 \cdot o_2, 2 \cdot o_1) \to o$



DERIVATIONS OF S

The set `Deriv` of rigid derivations is *coinductively* generated by:

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- Forget about the indexes: `S` collapses onto $\mathscr{D}$.

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  - During reduction, #5-`ax` will be replaced by P_5, even if there are other P_k concluded by $S = S_5$

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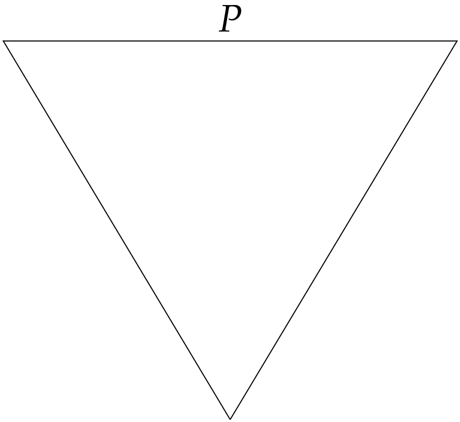

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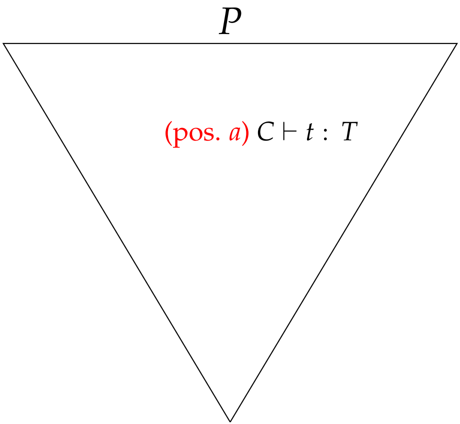

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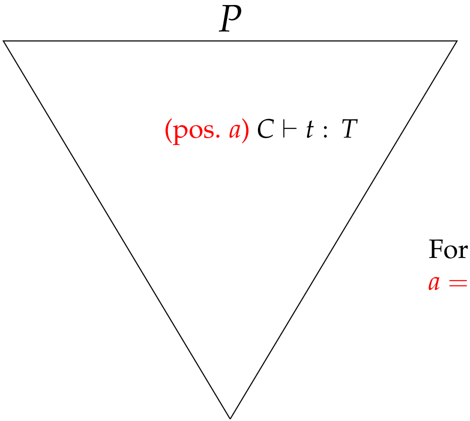

For instance
$a = 0 \cdot 1 \cdot 3 \cdot 0 \cdot 8 \cdot 1$

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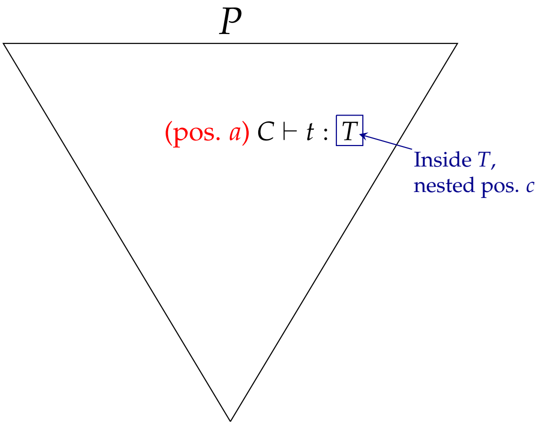

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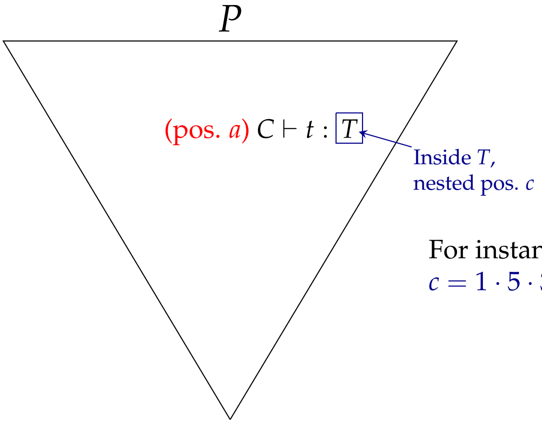

For instance
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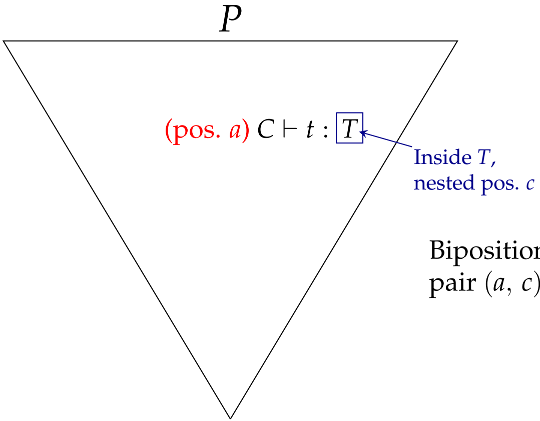

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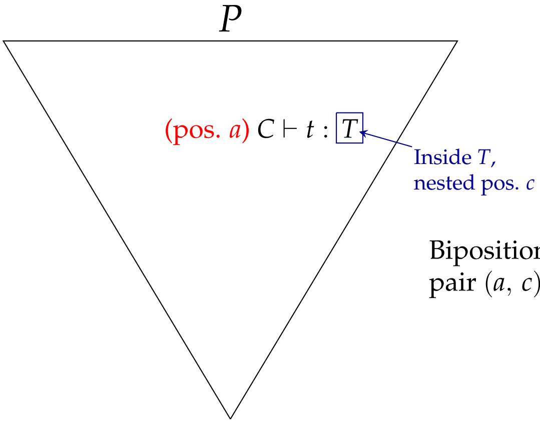

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Bisupport of P: the set of (right or left) bipositions

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Order, meet and join are given by the set-theoretic operations $\subseteq$, $\cap$, $\cup$ on bisupports.

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Argument 4: Subject expansion property holds for s.c.r.s. (assuming approximability only).

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- Every term is typable in $\mathtt{S}$. We say that $\mathtt{S}$ is **completely unsound**. In that case, since $\mathtt{S}$ enjoys SR and SE, $\mathtt{S}$ will provide us with a new model for pure **lambda-calculus**.

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Related works

- Jacopini[75]: **easy** terms (t is easy if it can be consistently equated to any other term)
- Berarducci[96]: **mute** terms ("The most undefined terms").
- Bucciarelli,Carraro,Favro,Salibra[15]: *Graph easy Sets of mute lambda terms*, TCS.

Relevance vs Irrelevance

- **Observation:** In system $\mathscr{R}$, $\lambda x.x$ (resp. $\lambda y.x$) can only be typed with a type of the form $[\tau] \to \tau$ (resp. $[\,] \to \tau$).

RELEVANCE VS IRRELEVANCE

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- In $\mathscr{R}_{\mathtt{w}}$, we may derive:

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  Proof.

$$\dfrac{\Gamma; x : [\rho]_\omega \vdash t : \rho}{\Gamma \vdash \lambda x.t : [\rho]_\omega \to \rho \ \ (= \rho)}\mathtt{abs}$$

$$\dfrac{\Gamma \vdash t : \rho \ \ (= [\rho]_\omega \to \rho) \qquad (\Gamma \vdash u : \rho)_\omega}{\Gamma \vdash t\, u : \rho}\mathtt{app}$$

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- Difficulty to see the typing constraints on x.

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We study then **typability** as a first order theory. For that, we will rather study typability in $\mathtt{S}$.
System $\mathtt{S}$ collapses on $\mathscr{R}$. Thus, if every term is typable in $\mathtt{S}$, then every term is typable in $\mathscr{R}$.

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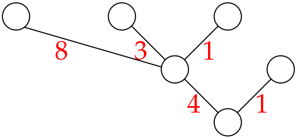
What is a correct type ?



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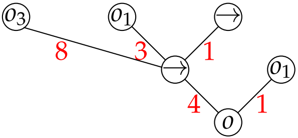

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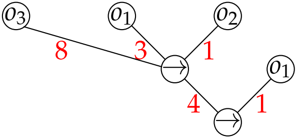

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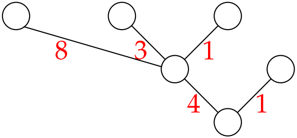

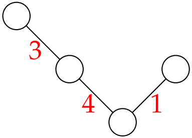

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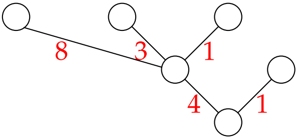

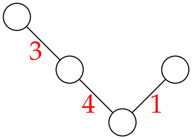

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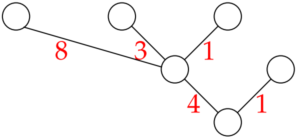

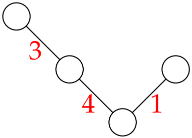

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Candidate Support: a set of positions that is the support of a type

- $c \to_{\mathtt{t1}} c\cdot k$ (a candidate supp is a tree)
- $c\cdot 1 \to_{\mathtt{t2}} c\cdot k$ (if a node does not have a 1-son, it is a leaf)

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- Then, we show that there is a *non-empty* set that satisfies them.

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- $(a,\, c) \to_{\mathtt{asc}} (a \cdot 1, 1 \cdot c)$ if $t(a) = @$.
- $(a,\, 1 \cdot c) \to (a \cdot 0,\, c)$ if $t(a) = \lambda x$.
- $(a, k \cdot c) \to_{\mathtt{pi}} (pos(k), c)$ if $t(a) = \lambda x$ and $k \in \mathtt{Tr}_1(a)$.
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Theorem: This yields a non-sensible model that discriminates terms according to their order.

Plan

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- Thus, the choice of types in axiom rules must ensure that we have a **syntactic equality** for every `app`-rule.
- Moreover, we must avoid track conflict in the contexts.

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- Type system $\mathtt{S}_\mathtt{h}$ is obtained from $\mathtt{S}$ by replacing the $\mathtt{app}$-rule by:

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- Easy to show that every $\mathscr{R}$-derivation is the collapse of a $\mathtt{S_h}$-derivation.

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- **Interfaces:**
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  - A **complete interface** is given by a family of full sequence type isomorphisms $\phi_a : (S_k)_{k \in K}(a) \to (S'_k)_{k \in K'}(a)$ when a ranges over the `app`-nodes of P.

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- An **operable derivation** is a hybrid derivation endowed with a complete interface (for each `app`-rule).

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Lemma

Let Π a $\mathscr{R}$-derivation typing t and a reduction sequence $\mathscr{R}$ (of length $\leqslant \omega$) and P a hybrid representative of Π.
Any reduction choice sequence along $\mathscr{R}$ can be built-in inside a complete interface for P.

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- Since each interface isomorphism of the reduced derivation is a residual an interface isomorphism, interface I_i can be lifted to P.

Plan

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- Commutation with interface isomorphisms of P_1 and P_2.

RELATED AND FUTURE WORK

- Quantitative types for $\lambda\mu$ (ongoing work with Delia Kesner) and an explicit classical calculus.
- Can infinitary Strong Normalization be characterized ?
- *Categorical Adaptation* of this framework (ongoing work with D. Mazza and L. Pellisier).
- Equational theory of the Model.
- Is the collapse of $\mathscr{R}$ onto $\mathscr{D}$ (idempotent intersection) also surjective ?

QUESTIONS

Thank you for your attention !

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Let us remind rules `ax` and `abs`:

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Look at S_7
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